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Minimal Superstring Theories

James Carlisle

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A Thesis presented for the degree of
Doctor of Philosophy



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February 2006



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Minimal Superstring Theories

James Carlisle

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Abstract

In this thesis we study the $(2, 4m)$ series of Type 0A minimal superstring theories, which are intimately related to the KdV integrable hierarchy. Chapters 1 to 3 constitute a comprehensive review of the relevant background material, including noncritical string theory, matrix models and integrable systems. In Chapter 4 we generalise the $(2, 4m)$ theories so as to include unoriented worldsheet contributions in the partition sum. This generalisation is tested against a known result in the literature. Chapter 5 explores the D-branes of the theories in more detail and, by studying the Bäcklund transformation of the KdV hierarchy, gives conclusive evidence that the parameter speculated to control the number of ZZ branes in the theory is indeed quantised. The FZZT brane partition function is studied, and it is shown that the effects of the boundary cosmological constant arise only in certain predictable forms. Chapter 6 examines the string theory interpretation of the negative KdV hierarchy, which naively relates to supercritical string theories living in greater than ten dimensions. The models are seen to have some non-trivial characteristic properties, indicating that they do indeed describe valid string theories, but it is unclear whether said theories are actually supercritical.

Declaration

The work in this thesis is based on research carried out at The Department of Mathematical Sciences and The Centre for Particle Theory, The University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification.

Chapters 1 to 3 of this thesis present a review of background material for which no claim of originality is made. Chapters 4 to 6 are original work, some of which was done in collaboration with Professor Clifford V. Johnson and Jeffrey S. Pennington. Much of the content in these latter three chapters has appeared elsewhere:

- J. Carlisle and C.V. Johnson, *Unoriented Minimal Type 0 Strings*, JHEP, **12** (2004) 073, <http://arXiv.org/hep-th/0408159>. (Chapter 4 of this thesis)
- J. Carlisle, C.V. Johnson and J.S. Pennington, *Bäcklund Transformations, D-branes, and Fluxes in Minimal Type 0 Strings*, <http://arXiv.org/hep-th/0501006>. (Chapter 5 of this thesis)
- J. Carlisle, C.V. Johnson and J.S. Pennington, *D-branes and Fluxes as Bound States in Supersymmetric Quantum Mechanics*, <http://arXiv.org/hep-th/0511002>. (Chapter 5 of this thesis)
- J. Carlisle, *Exactly Solvable Supercritical String Theories?* <http://arXiv.org/hep-th/0511240>. (Chapter 6 of this thesis)

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Chapter 1

Introduction

1.1 Overview and Motivation

At the present point in time, (super)string theory is the best hope to unify all the aspects of nature in a single underlying mathematical framework. String theory, whose basic degree of freedom is the one-dimensional *string*, has none of the short-distance divergences that plague standard quantum field theories of point particles. In addition, string theory is the only consistent quantum theory of gravity that has been developed. These achievements are only tempered by the unphysical number of dimensions in which the theory is expected to exist (ten or twenty-six). However, attempts to surmount this problem and understand string theory in greater detail have led to the discovery of both physical and mathematical phenomena that were undreamt of twenty-five years ago. String theory is a highly elegant structure and, even if it turns out to have no physical significance, is worth studying in its own right.

One of the remarkable achievements of string theorists has been the discovery of dualities and correspondences that relate the different string theories both amongst themselves, and to other theories such as conformal field theories and matrix models. These dualities demonstrate to us that the mathematics underpinning the theory has much more structure than it was initially given credit for. Specifically, many apparently different theories actually turn out to be different manifestations of the same theory. The most famous example of this is the mysterious M-theory, that is believed to unite all the ten dimensional superstring models. The existence of these dualities is at least as fascinating as the string theories themselves, and there seems to be a great deal that they can teach us. Much of the progress on dualities in string



theory has come hand-in-hand with the discovery of D-branes. These objects are as fundamental to the theory as are the strings themselves, and have dominated the subject in the preceding ten years or so.

In this thesis, we will study a particular set of superstring theories that do not exist in the critical dimensionality of ten. These are *noncritical* superstring theories. Of particular interest will be the *minimal* string theories, which have a finite number of conformal primary operators on the worldsheet. These theories are simple enough to be tractable, yet complicated enough to exhibit highly non-trivial phenomena. As such they serve as interesting ‘toy models’, with the potential to teach us a lot about more realistic string theories.

We will see that these theories have correspondences that relate them to both matrix models and integrable systems. In particular, a certain set of Type 0A minimal superstring theories, living in less than two target spacetime dimensions, will be seen to be underpinned by the Korteweg-de Vries (KdV) integrable hierarchy. This connection allows one to study so-called ‘string equations’, which in principle allow *exact* solutions to be obtained for the partition functions of the theories. The connection between certain string theories and integrable systems is fascinating, and time after time we will see how this correspondence keeps on growing as new results from the vast integrable literature are translated into a string theory setting. We will find that integrable models serve to motivate a lot of the results presented below.

The outline of this thesis is as follows. The remainder of Chapter 1 will be a general introduction to the theory of relativistic quantum strings. In this chapter we will explore the basics of string quantisation and the spectrum of the theory. We will then introduce the notion of D-branes before moving on to study the noncritical string theories that will be relevant later. This, like all of the three introductory chapters, will be quite comprehensive, with the justification that there is a lot of background material underpinning the formalism of later chapters that is vital for a complete understanding. Material intended for these chapters that is not essential for an understanding of later results has been moved to the appendices whenever possible. In particular, basic introductions to critical string theory and conformal field theory can be found in Appendix A, with instead only short summaries¹ of these topics in

¹Aimed purely to refresh the memory of the reader.

Chapter 1. This is done to condense the introduction for the more experienced reader, whilst still providing a self-contained thesis for those with less prior knowledge.

Chapter 2 explores the formalism of integrable systems that will be relied upon heavily later on. The main focus of this chapter will be the KdV hierarchy and its underlying structure. This will lead us to important results such as the Miura map relating the KdV and modified KdV (mKdV) hierarchies, and Bäcklund transformations. Both of these results, and many others, will later reappear in a string theory setting.

Chapter 3 studies minimal string and superstring theories, their D-branes, and their underlying conformal field theories. These string theories will be related to matrix models via discretisation of the string worldsheet. It will be shown how string equations can be obtained by taking special *double scaling* limits of these models. This will lead to an underlying integrable structure, which will begin to tie in with the results of Chapter 2.

Chapters 4 to 6 consist of mostly new results. Each chapter will be self-contained and include its own summary and discussion. Chapter 4 will summarise how the bosonic minimal string theories were originally generalised to include non-orientable worldsheets. This will be followed by new work conjecturing the analogous generalisation of certain Type 0A minimal superstring theories. The consequences of this conjecture will be studied in detail, and the outcome tested against known results.

Chapter 5 will study more aspects of minimal superstrings. Firstly, by identifying the Bäcklund transformation of the string equations, we will show that an important parameter, Γ , is forced to be quantised. This is consistent with its interpretation as some number of background ZZ branes/RR flux units. We will then study the partition functions of FZZT branes as they probe the background. This will tie in nicely with the existence of the Bäcklund transformation and bring across even more integrable systems formalism, such as a connection with supersymmetric quantum mechanics. Finally, we will explore the FZZT partition function for non-zero boundary cosmological constant. We will demonstrate that the effects of this constant arise only in a certain number of predictable forms, and we will go a long way towards deriving rules that allow us to determine these in general.

Chapter 6 will again delve into the integrable literature from a string perspective. By studying the so-called *negative KdV* hierarchy, we will speculate a natural (formal) analytical continuation of the earlier string equations that would naively correspond

to string theories living in *greater than*, rather than less than, the critical number of dimensions. We will analyse the mathematics of the new string equations and show that they do make sense as string theories. However, these *supercritical* theories are expected to be unstable and very little is known about their physics. Consequently, it is difficult to say whether the negative KdV hierarchy really does underpin some aspect of them. Therefore the main thrust of this chapter will be the mathematics itself, which is novel and interesting in its own right. At the end of the chapter we will tentatively try to describe the possible string physics and we will find that the theory does have some of the salient features expected of a supercritical theory, but this is far from conclusive.

Now that this overview is complete, we will start with the very basics of string theory. As mentioned above, Sections 1.2 and 1.3 can be found in much more extensive detail in Appendix A. In this chapter we present mere summaries.

1.2 What Is String Theory?

The basic one-dimensional object in string theory, the string, will sweep out a two-dimensional *worldsheet* in flat Minkowski spacetime, $X^\mu(\sigma, \tau)$,² parametrised in terms of the *worldsheet coordinates* $\sigma = 0, \dots, l$ and $\tau \in \mathbb{R}$. For open strings, $X^\mu(0, \tau)$ and $X^\mu(l, \tau)$ define the two ends of the string in spacetime, whereas for closed strings σ is defined to be periodic with $X^\mu(0, \tau) = X^\mu(l, \tau)$.³ In the literature, the X^μ are referred to as the *target spacetime* coordinates, whereas τ and σ are the *worldsheet coordinates*. In this regard, string theory can be thought of as a conformal field theory, with fields X^μ , living on the two dimensional worldsheet. This interpretation will be returned to later.

Basic string theory can be described by the *Polyakov* action [1]:

$$S_{Pol} = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu, \quad (1.1)$$

in terms of the (*Lorentzian*) *worldsheet metric* $\gamma_{ab}(\tau, \sigma)$, with $\gamma \equiv \det \gamma_{ab}$. The Polyakov action, (1.1), is classically invariant under various transformations. The

² X^0 plays the role of time.

³Accordingly, open string worldsheets have boundaries, whereas closed string worldsheets do not.

first of these is known as worldsheet *diffeomorphism invariance*, which is invariance under reparameterisations, $\sigma' = \sigma'(\sigma, \tau)$, $\tau' = \tau'(\sigma, \tau)$, on the worldsheet:

$$\begin{aligned} X^\mu(\tau, \sigma) &\mapsto X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma), \\ \gamma_{ab}(\tau, \sigma) &\mapsto \gamma'_{ab}(\tau', \sigma'), \quad \text{with} \quad \frac{\partial \sigma'^c}{\partial \sigma^a} \frac{\partial \sigma'^d}{\partial \sigma^b} \gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma). \end{aligned} \quad (1.2)$$

This invariance encodes the fact that the physics should be independent of the worldsheet coordinates. We also have *Weyl invariance*:

$$\begin{aligned} X^\mu(\tau, \sigma) &\mapsto X'^\mu(\tau, \sigma) = X^\mu(\tau, \sigma), \\ \gamma_{ab}(\tau, \sigma) &\mapsto \gamma'_{ab}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \gamma_{ab}(\tau, \sigma), \end{aligned} \quad (1.3)$$

where $\omega(\tau, \sigma)$ is arbitrary. Using the diffeomorphism and Weyl invariances we have exactly enough freedom to fix the three independent components of the worldsheet metric γ_{ab} in the Polyakov action (1.1).

One can quantise the Polyakov action in a number of ways. In the simplest, known as *light-cone quantisation*, we define $X^\pm \equiv (X^0 \pm X^1)/\sqrt{2}$ and use up the diffeomorphism and Weyl freedom of the metric to choose:

$$X^+ = \tau, \quad \partial_\sigma \gamma_{\sigma\sigma} = 0, \quad \det \gamma_{ab} = -1, \quad (1.4)$$

This is known as the *light-cone gauge*, and it has the advantage that we have fixed all the gauge freedom in the metric. The disadvantage is that we have lost the manifest covariance of the theory since we have picked out some of the target spacetime dimensions as in some way special.

For open strings we impose Neumann boundary conditions. The equations of motion can be solved to give:

$$X^i(\tau, \sigma) = x^i + \frac{p_i \tau}{p^+} + i(2\alpha')^{1/2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \alpha_n^i e^{-\pi i n c \tau / l} \cos \frac{\pi n \sigma}{l}, \quad (1.5)$$

where $c = l/(2\pi\alpha'p^+)$. The x^i and p^i are constants (representing the overall centre-of-mass motion of the string), as are the $\alpha_n^i = (\alpha_{-n}^i)^\dagger$ (which relate to the vibration modes). Quantisation yields the following commutators:

$$[x^-, p^+] = -i, \quad [x^i, p^j] = \delta^{ij}, \quad [\alpha_m^i, \alpha_n^j] = m \delta^{ij} \delta_{m, -n}. \quad (1.6)$$

The operators α_n^i have become creation ($n < 0$) and annihilation ($n > 0$) operators. The vacuum state of the theory is then $|0; k\rangle$, with centre-of-mass momenta $k =$

(k^+, k^i) (k^+ is like the energy of the string state) such that:

$$p^+|0; k\rangle = k^+|0; k\rangle, \quad p^i|0; k\rangle = k^i|0; k\rangle, \quad \alpha_n^i|0; k\rangle = 0, \quad m > 0. \quad (1.7)$$

A general string state can be constructed from $|0; k\rangle$ by acting on it with the α_{-n}^i .

$$|N; k\rangle \propto \left[\prod_{i=2}^{D-1} \prod_{n=1}^{\infty} (\alpha_{-n}^i)^{N_{i,n}} \right] |0; k\rangle, \quad (1.8)$$

where $N_{i,n}$ is the occupation number for the mode (i, n) , $i = 2, \dots, D-1$ and $n = 1, 2, \dots$. Each combination of the α_{-n}^i produces a different ‘particle’ in the target space. However, as we have already mentioned, quantisation by this method has obscured Lorentz covariance. It turns out that for the theory to still be Lorentz invariant after quantisation, we are forced to choose $D = 26$.

It turns out that the lightest open string state, the scalar $|0; k\rangle$, has negative mass squared and so is a *tachyon*. This means that the vacuum of the theory is unstable.

The simplest excited state is just $\alpha_{-1}^i|0; k\rangle$, which has zero mass. This has exactly the $D-2$ components expected of a massless vector particle.

In the closed string case things are much the same, except that there are now two sets of operators β_n^i and $\tilde{\beta}_n^i$, corresponding to modes of vibration moving in opposite directions around the string. These are referred to as *left-movers* and *right-movers* respectively. Imposing invariance under σ -translations on the states gives the *level-matching* condition that the number of left-movers excited must be the same as the number of right-movers. The lightest excited state is again massless. It is a rank two tensor, and will actually turn out to be the *graviton* of the string theory.

The light-cone gauge quantisation explained above is the simplest method of quantising the string theory. Far more elegant is to use *path integral quantisation*, which proceeds in exactly the same way as in ordinary quantum field theory. We define the string partition function:

$$Z = \int \mathcal{D}X \mathcal{D}g e^{-S}, \quad (1.9)$$

where we have performed functional integration over the ‘fields’ X and g .⁴

⁴Here we have Wick-rotated to a Euclidean worldsheet of signature $(+, +)$ with $\tau \mapsto i\sigma^1$, $\sigma \mapsto \sigma^2$. Consequently, g_{ab} is defined to be the Euclidean version of the Lorentzian worldsheet metric γ_{ab} used previously.

So far we have been considering a single string at $X^0 = -\infty$ propagating to $X^0 = \infty$. We can of course generalise the above to include string interactions, which can be visualised as the splitting and joining of strings. This means introducing an implicit sum over worldsheet topologies into $\mathcal{D}g$. Specifically we only consider *vacuum diagrams*, which are worldsheets without external legs. Just like in ordinary quantum field theory, the diagrams with external legs, the n -point functions, can be derived once we know the vacuum partition function itself. In string theory, the insertion of initial and final states corresponds to the insertion of so-called *vertex operators* onto these vacuum worldsheets.

We can add an extra term into the action, S_χ , that will control the strength of the string interactions:

$$S_\chi = \frac{\lambda}{4\pi} \int_M d^2\sigma g^{1/2} R + \frac{\lambda}{2\pi} \int_{\partial M} ds k, \quad (1.10)$$

where λ is a constant, M denotes the worldsheet, and ∂M its boundary. Here, R is the two dimensional worldsheet Ricci scalar, ds is the proper time along the boundary, and k is the curvature of the boundary. This latter curvature term is of course only present in the case of open strings.

Since the worldsheet is two-dimensional, we find that the term involving R is easy to evaluate. It is just λ multiplied the *Euler number*, χ , of the worldsheet. It is given by:

$$\chi = 2 - 2h - b, \quad (1.11)$$

where h is the number of handles on the worldsheet and b is the number of boundaries. We now see that e^λ is a coupling that controls the strength of the splitting and joining of worldsheets. Adding an extra handle to the worldsheet weights the partition function by $e^{2\lambda} \equiv g_s^2$. Adding an extra boundary weights it by $e^\lambda \equiv g_s$. These processes are the string analogues of 1-loop Feynman diagrams, encoding the emission and absorption of virtual closed and open strings respectively. We therefore see that the coupling associated with closed strings is the square of that associated with open strings.

The partition function formulation of string theory is extremely powerful. However, to really understand the full power of the formalism we will need to study some basic conformal field theory.

1.3 Conformal Field Theory and Vertex Operators in String Theory

A conformal field theory (CFT) in two dimensions is defined by an action, $S[X^\mu]$, in terms of a set of fields $X^\mu(\sigma^1, \sigma^2)$, where σ^1 and σ^2 are Euclidean worldsheet coordinates with flat metric δ_{ab} . It is conventional to use complex coordinates with $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$. We have:

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \quad (1.12)$$

A typical action is:

$$S_f = \frac{1}{2\pi\alpha'} \int d^2z \partial_1 X^\mu \bar{\partial} X_\mu. \quad (1.13)$$

An important property of a CFT is *conformal invariance*. This is an invariance under coordinate transformations, given infinitesimally by $z \mapsto z' = z + \varepsilon v(z)$. This corresponds to the following finite transformation:

$$X^\mu(z, \bar{z}) \mapsto X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}), \quad z' = f(z), \quad (1.14)$$

for any holomorphic $f(z)$. If an action is invariant under worldsheet translations then there will be a conserved Noether current, $j_a = iv^b T_{ab}$, associated with it. T_{ab} is known as the *energy-momentum tensor*, and is traceless in theories with full conformal invariance. This implies that $T_{zz} = T(z)$ is holomorphic and $T_{\bar{z}\bar{z}} = \bar{T}(z)$ is antiholomorphic.

We calculate expectation values of quantities in a conformal field theory via the path integral:

$$\langle \mathcal{F}[X] \rangle = \int \mathcal{D}X e^{-S} \mathcal{F}[X]. \quad (1.15)$$

Key ideas will be the concepts of *local operators* and the *operator product expansion*. Associated to any conformal field theory is a *complete set* of operators $\{\mathcal{A}_j(z, \bar{z})\}$, the expectation values of which are calculated via (1.15). The set of operators will

include the identity operator and co-ordinate derivatives of each field involved in the action. It is important to evaluate *correlation functions*, which are expectation values of products of operators:

$$\langle \mathcal{A}_{i_1}(z_1, \bar{z}_1) \mathcal{A}_{i_2}(z_2, \bar{z}_2) \cdots \mathcal{A}_{i_n}(z_n, \bar{z}_n) \rangle. \quad (1.16)$$

Since the set of operators is complete, we can expand any operator product into a linear sum of other operators. Conformal invariance puts special constraints upon the form of this *operator product expansion*. Any product of operators with components of T_{ab} will be especially simple. We can always choose a basis of operators $\{\mathcal{A}_j(z, \bar{z})\}$ that transform in a certain way under a rigid transformation, $z \mapsto z' = az$, for some complex parameter a . This transformation serves to rotate and rescale the coordinate system (but not deform it otherwise). We have:

$$\mathcal{A}_j(z, \bar{z}) \mapsto \mathcal{A}'_j(z', \bar{z}') = a^{-h_j} \bar{a}^{-\bar{h}_j} \mathcal{A}_j(z, \bar{z}), \quad (1.17)$$

where (h_j, \bar{h}_j) are known as the *weights* of \mathcal{A}_j . The sum $h_j + \bar{h}_j$ is known as the *dimension* of \mathcal{A}_j , determining its behaviour under scaling, and the difference $h_j - \bar{h}_j$ is the *spin*, determining the behaviour under rotations. In what follows we will often suppress the \bar{z} argument of fields, which we can do without loss of generality because the same arguments apply for \bar{z} as apply for z .

It turns out that the spectrum of weights, h_j , in any two-dimensional CFT consists of the infinite-integer spaced series:

$$h_j^{[k]} = \Delta_j + k, \quad k = 0, 1, 2, \dots, \quad (1.18)$$

where Δ_j is the minimum weight in each series. So, we have towers of states all related to the operators with lowest weights, Δ_j . These special operators, \mathcal{O}_j , are known as *primary fields* or *tensor operators*. Under general conformal transformation they transform as:

$$\mathcal{O}_j(z, \bar{z}) \mapsto \mathcal{O}'_j(z', \bar{z}') = (\partial_z z')^{-h_j} (\partial_{\bar{z}} \bar{z}')^{-\bar{h}_j} \mathcal{O}_j(z, \bar{z}). \quad (1.19)$$

In general, a theory will have some number of these primary fields \mathcal{O}_j with weights $(\Delta_j, \tilde{\Delta}_j)$. Derived from these will be *conformal families* $[\mathcal{O}_j]$ of *secondary operators* with weights $(\Delta_j + k, \tilde{\Delta}_j + \tilde{k})$ for $k = 1, 2, \dots$. Under conformal transformations, each member of a conformal family transforms only in terms of members of the same family. There is no mixing between different families, so each family forms some irreducible

representation of the conformal algebra. The sum of all the conformal families will constitute the complete set of operators \mathcal{A}_j .

In practice, the operators, \mathcal{A}_j , will be constructed from various combinations of the X^μ . Some typical examples, along with their weights, can be found in Section A.2. One particularly useful operator is $:e^{ik \cdot X}:$, which is an example of a *vertex operator* in the string theory context. Here the $::$ represent conformal normal ordering. Insertion of this vertex operator into the path integral (1.15) is equivalent to puncturing the worldsheet at its position (z, \bar{z}) . For instance, the basic closed string worldsheet, the infinite cylinder, is conformally equivalent to a twice-punctured sphere. We can think of these punctures as specifying the initial and final states of the string. So, if we were to insert the vertex operator $:e^{ik \cdot X}:$ onto the disc, we should interpret this as the preparation of an initial tachyon state with momentum k . Vertex operators corresponding to other particles in the closed string spectrum have the form $: \mathcal{P}(\partial X, \partial^2 X, \dots; \bar{\partial} X, \bar{\partial}^2 X, \dots) e^{ik \cdot X} :$. Similarly, it is easy to see that initial and final open string states can be thought of as the insertion of operators on worldsheet boundaries: so an infinite strip representing propagation of a single open string can be thought of as a disc with two boundary insertions. Correspondingly we can separate the action into a *bulk* theory and a *boundary* theory.

This mapping between states of the string theory and the basic operators is known as the *state-operator correspondence*. It can be shown to be bijective. To examine this state-operator mapping in more detail we expand the $T(z)$ and $\tilde{T}(\bar{z})$ components of the energy-momentum tensor as:

$$T(z) = \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}}, \quad \tilde{T}(\bar{z}) = \sum_{m \in \mathbb{Z}} \frac{\tilde{L}_m}{\bar{z}^{m+2}}. \quad (1.20)$$

The coefficients L_m and \tilde{L}_m are known as *Virasoro generators*. They satisfy the *Virasoro algebra*:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \\ [\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{\tilde{c}}{12}(m^3 - m)\delta_{m,-n}, \end{aligned} \quad (1.21)$$

where the constants c and \tilde{c} are known as central charges. For most common theories $c = \tilde{c}$. The specific value of the central charge depends on the particular form of the

conformal field theory action. In the case of free scalars X^μ , as in (1.13), the central charge is equal to D , the number of such fields (i.e. each scalar contributes unity to the central charge). Free fermions contribute a half to the central charge.

Using the state-operator mapping we see that to the primary fields we can associate *primary states*, also known as vectors:

$$|j\rangle = \mathcal{O}_j(0)|0\rangle, \quad (1.22)$$

where $|0\rangle$ is the vacuum of the theory, which is the state corresponding to the identity operator. A primary state is also known as a *highest weight state* because it is annihilated by all lowering operators L_m with $m > 0$. It also satisfies $L_0|j\rangle = \Delta_j|j\rangle$. Secondary fields in the associated conformal family can be generated by repeated action of the Virasoro generators. This means that all information about the CFT is contained in the correlators of primary fields.

The Virasoro generators play a crucial role in string theory. For instance, in closed string theory the L_m are given in terms of sums of products of the β_n^μ :

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \beta_{m-n}^\mu \beta_{\mu n} :, \quad (1.23)$$

and similarly for \tilde{L}_m . Here $::$ refers to the usual creation-annihilation normal ordering: placing lowering operators to the right of raising operators.

Finally in this section, let it be stated that a conformal field theory with a positive definite inner product between states in its Hilbert space is known as a *unitary CFT*. These CFTs must have central charges that are greater than or equal to zero.

1.4 The Weyl Anomaly

We earlier attempted to quantise the Polyakov action in the light-cone gauge. This was insightful, but somewhat lost the manifest covariance of the string theory. It turns out to be possible to retain this covariance using the method of *BRST quantisation*. This involves the *Faddeev-Popov* ghost method of quantum field theory. One starts with the path integral formulation with partition function, \hat{Z} :

$$\hat{Z} = \int \mathcal{D}X \mathcal{D}g e^{-S_M[X,g]}, \quad (1.24)$$

for an action S_M such as S_{Pol} in (1.1).

Clearly the partition function \hat{Z} has the problem that the integral overcounts physically equivalent configurations. By this we mean that all gauge-equivalent configurations (i.e. configurations that can be related by diffeomorphism and Weyl invariance) should be considered as a single configuration physically. Given a particular configuration, the set of all possible gauge-equivalent configurations is known as a *gauge orbit*. To properly define the theory we need to choose a *gauge slice* that encompasses exactly one configuration in each gauge orbit. The ‘true’ partition function is therefore given by:

$$Z = \int \frac{\mathcal{D}X \mathcal{D}g}{V_{\text{diff} \times \text{Weyl}}} e^{-S_M[X,g]}, \quad (1.25)$$

where $V_{\text{diff} \times \text{Weyl}}$ is the ‘volume’ of the particular gauge orbit. A short derivation then leads to the relation:

$$Z[\hat{g}] = \int \mathcal{D}X \Delta_{FP}(\hat{g}) e^{-S_M[X,\hat{g}]}, \quad (1.26)$$

where in the process we have used the gauge freedom to fix the metric at some specific functional form, \hat{g} , which we are free to choose. The function $\Delta_{FP}(\hat{g})$ is the so-called *Faddeev-Popov* determinant given by:

$$\Delta_{FP}(\hat{g}) = \int \mathcal{D}[ghosts] e^{-S_g}. \quad (1.27)$$

Here we have introduced *ghost* fields⁵. The specific form of the ghost action, S_g , for string theory in the case of a conformally flat metric⁶, $\hat{g}_{ab} = e^{\omega(\sigma)} \delta_{ab}$, is:

$$S_g = \frac{1}{2\pi} \int d^2z \{ b_{zz} \bar{\partial} c^z + b_{\bar{z}\bar{z}} \partial c^{\bar{z}} \}, \quad (1.28)$$

again using complex worldsheet coordinates. So, we have eliminated the gauge invariance at the cost of introducing these new ghost fields, b and c ; which are anticommuting and have their own creation-annihilation operators and oscillator modes. b_{zz} ($b_{\bar{z}\bar{z}}$) is a tensor field with weights $(2, 0)$ ($(0, 2)$); c^z ($c^{\bar{z}}$) is a tensor field with weights $(-1, 0)$ ($(0, -1)$). These weights imply that S_g will be conformally invariant, because the combined weight of integrand multiplied by d^2z is $(0, 0)$. We can therefore re-write the partition function as:

$$Z = \int \mathcal{D}X \mathcal{D}b \mathcal{D}c \mathcal{D}\bar{b} \mathcal{D}\bar{c} e^{-S_M - S_g}. \quad (1.29)$$

⁵So-called because they only appear in closed loops rather than as external particles.

⁶The *conformal gauge*.

Now that we have introduced the ghosts to eliminate the gauge redundancy, we can calculate the change in the partition function under an infinitesimal Weyl transformation $\omega \rightarrow \omega + \delta\omega$:

$$\delta_{Weyl} \ln Z = \frac{c_{matter} - 26}{24\pi} \int d^2\sigma \sqrt{g} R \delta\omega, \quad (1.30)$$

where c_{matter} is the central charge of the matter conformal field theory with action S_M . So, for the quantised theory to be Weyl invariant we require that $c_{matter} = 26$. That is, we require that the central charge of the matter sector cancel with the central charge, -26 , of the ghosts. Otherwise the Weyl invariance of the classical theory is anomalously broken. In the case of the Polyakov action, S_{Pol} , we have $c_{matter} = D$, which implies we must have $D = 26$: *critical string theory*. We have seen this condition before when we looked at light-cone gauge quantisation.

One may ask whether the lack of Weyl invariance matters in string theory: can we not just choose $\omega(\sigma)$? The answer is no, because that would cause other problems for us. For instance, choosing $\omega(\sigma) = 0$ gives a theory that is not invariant under diffeomorphism transformations. The only other possibility is to treat $\omega(\sigma)$ as *a field in its own right!* We consider a field theory in which we only have to fix two of the three independent metric components, but with the extra field $\omega(\sigma)$. So $\omega(\sigma)$ is like an *extra dimension*. In standard conventions we retain the function $\omega(\sigma)$ for the *fake Weyl invariance* explained below, and instead use the coordinate invariance to put the metric into the form:

$$g_{ab}(\sigma) = e^{\varphi(\sigma)} \hat{g}_{ab}(\sigma), \quad (1.31)$$

where $\varphi(\sigma)$ is the Weyl field, known as the *Liouville mode*, to be integrated over in the partition function. We then have a theory with the following fake Weyl invariance for *any* value of the central charge:

$$\hat{g}_{ab}(\sigma) \mapsto e^{\omega(\sigma)} \hat{g}_{ab}(\sigma), \quad \varphi(\sigma) \mapsto \varphi(\sigma) - \omega(\sigma). \quad (1.32)$$

So it is as if \hat{g}_{ab} were the true metric with $\varphi(\sigma)$ just another field. This is known as *noncritical string theory* (see Section 1.8).

1.5 More General String Theories

The most general coordinate invariant action with two derivatives is the *non-linear sigma model*:

$$S_\Sigma = -\frac{1}{4\pi\alpha'} \int d^2\sigma g^{1/2} \left\{ (g^{ab}G_{\mu\nu}(X) + i\varepsilon^{ab}B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X) \right\}, \quad (1.33)$$

where ε^{ab} is the usual antisymmetric tensor. The functions $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ are known as the *spacetime metric*, the *antisymmetric tensor* and the *dilaton* respectively. We can think of these functions as specifying a spacetime background in which the strings live. Weyl invariance requires them to satisfy (at order α'):

$$\alpha' \mathbb{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} = 0, \quad (1.34)$$

$$-\frac{\alpha'}{2} \nabla^\kappa H_{\kappa\mu\nu} + \alpha' \nabla^\kappa \Phi H_{\kappa\mu\nu} = 0, \quad (1.35)$$

$$\frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0, \quad (1.36)$$

where $\mathbb{R}_{\mu\nu}$ is the spacetime Ricci tensor (written as \mathbb{R} to distinguish it from its world-sheet counterpart); ∇_μ is the usual covariant derivative; and $H_{\mu\nu\lambda}$ is the three-form field strength associated to $B_{\mu\nu}$:

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu}. \quad (1.37)$$

In the case of critical String theory with $B_{\mu\nu} = 0$ and $\Phi = 0$ we find that (1.34) reduces to $\mathbb{R}_{\mu\nu} = 0$, which is *Einstein's equation*! So we have obtained general relativity for free. Only spacetime metrics satisfying Einstein's equation are permitted, at least to leading order in α' . Notice also that in the case $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$ and $\Phi = \Phi_0$, which satisfies the Weyl constraints for any constant Φ_0 , then the action reduces to that previously studied in (1.10) with $\lambda = \Phi_0$. So we have that the string coupling is just the exponential of the dilaton, $g_s = e^{\Phi_0}$. We therefore see that g_s is no longer a free parameter.

From a physical point of view one wonders what the interpretation of our choice of background is. It turns out that this background is just a *coherent state of strings* [1]. That is, it can be obtained by exponentiating string vertex operators (see Section 1.3). For instance, the spacetime metric $G_{\mu\nu}(X)$ is just a coherent state of the first closed string excitation: the massless tensor. We now see that this is justly interpreted as *the*

graviton. So string theory is proving to be far richer than we had at first anticipated: different backgrounds correspond to different states in the same basic theory.

So far we have only studied bosonic string theory, which, as we have seen, has the problem that its spectrum contains a tachyon. To rectify this problem the superstring was studied; we will review this in the next section. However, it would be remiss not to mention some further complications that arise when quantising string theory. This discussion can be found in Section A.3.

1.6 Superstring Theories

So far we have only discussed bosonic string theory. As we have seen, this has the serious problem of a tachyonic ground state⁷. In an attempt to surmount these problems, supersymmetric string theories were studied. In this section we will briefly summarise some of the salient points of these superstrings.

The basic idea is to add anticommuting worldsheet fermions, $\psi(z)$ and $\tilde{\psi}(\bar{z})$, into the action. Since ψ is holomorphic and $\tilde{\psi}$ antiholomorphic, these correspond to left and right moving fields respectively. The action in (1.13) then becomes:

$$S = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial_1 X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right) \quad (1.38)$$

This defines a *superconformal* field theory living on the string worldsheet. Each fermion contributes a half to the total central charge and so correspondingly we can write:

$$c = \left(1 + \frac{1}{2} \right) D = \frac{3}{2} D \equiv \frac{3}{2} \hat{c} \quad (1.39)$$

We can quantise the theory in the same way that we quantised the bosonic string. This time, Weyl invariance implies that we must choose $D = 10$. There is an added complication though. For closed strings we again impose the boundary condition $X^\mu(\sigma_1 + 2\pi) = X^\mu(\sigma_1)$, which encodes the fact that we have moved once around the string. In the case of the fermionic fields we have slightly more freedom than this; we

⁷Though with recent developments in understanding tachyon condensation processes, this may not be as much of a problem as was at first envisioned.

can also make $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\bar{z})$ antiperiodic if we so desire. We have:

$$\psi(z + 2\pi) = +\psi(z) \quad \text{Ramond (R)} \quad (1.40)$$

$$\psi(z + 2\pi) = -\psi(z) \quad \text{Neveu-Schwarz (NS)} \quad (1.41)$$

and similarly for $\tilde{\psi}^\mu(\bar{z})$. We also label states as ‘+’ or ‘−’, according to the value of the *worldsheet fermion number*, which we will not discuss here. So in the case of closed strings we have sixteen possible sectors, denoted (R+, R+), (R+, R−), etc. In the case of open strings the analogous periodicity/antiperiodicity condition is slightly different, but it amounts to there being four possible sectors: R+; R−; NS+; and NS−.

The spectrum of the model can be found in an analogous way to that of the bosonic string. For closed strings one finds that the NS-NS sectors contain spacetime bosonic fields corresponding to the metric, $G_{\mu\nu}$, antisymmetric tensor, $B_{\mu\nu}$, and dilaton, Φ . The R-NS and NS-R sectors contain spacetime fermions such as the gravitino. The R-R sectors also contain spacetime bosonic excitations. These correspond to n -form potentials.

However, in order to restrict to physically consistent string theories, with key properties such as modular invariance, we must only choose certain subsets of the sectors on offer. This is known as the *GSO* projection. There turn out to be five consistent superstring theories without tachyons. These are Type I (open plus closed, unoriented), Types IIA (closed, nonchiral in spacetime) and IIB (closed, chiral in spacetime), and two closed heterotic string theories⁸. Whilst it seems as if all these theories are distinct, it turns out that they are all related via a series of dualities.

As well as the tachyon free theories discussed in the preceding paragraph, there are also two other consistent superstring theories known as Type 0A and Type 0B. They have the following spectra:

$$0A : \quad (\text{NS}+, \text{NS}+) (\text{NS}-, \text{NS}-) (\text{R}+, \text{R}-) (\text{R}-, \text{R}+), \quad (1.42)$$

$$0B : \quad (\text{NS}+, \text{NS}+) (\text{NS}-, \text{NS}-) (\text{R}+, \text{R}+) (\text{R}-, \text{R}-). \quad (1.43)$$

Both of these theories have only bosonic closed string spacetime excitations, despite being supersymmetric on the worldsheet. Both theories also contain the tachyon. It

⁸These are clever combinations of left-moving bosonic strings and right-moving superstrings.

is mainly the Type 0 theories that will be studied in this thesis.

1.7 D-branes

We earlier imposed Neumann boundary conditions on the endpoints of the open string⁹. However, another way to satisfy the equation of motion of the endpoints is to impose *Dirichlet* boundary conditions upon them. These boundary conditions fix some of the X^μ to particular values at the endpoints. Specifically, if we impose Neumann boundary conditions on p of the D space coordinates, and Dirichlet conditions on $D-p$ directions, then this is equivalent to fixing the ends of the strings to lying on a p dimensional hyperplane known as a D-brane, or Dp -brane in this case. This manifestly breaks some of the Poincaré invariance of the theory, but turns out to be acceptable, and even necessary.

It turns out that the D-branes are dynamical objects in their own right. That they are a vital ingredient of string theories can be seen in many ways, such as when one tries to compactify the theories. Not only must open strings end on them, but also they source some of the closed string R-R fields in superstring theories. The Type IIA theory, for instance, has 1-form, 3-form, 5-form, 7-form and 9-form potentials. These are correspondingly sourced by D0, D2, D4, D6 and D8-branes respectively. We earlier remarked that the Type I string theory had both open and closed strings, whereas the Type II theories had exclusively closed strings. We now see that this is only true before we introduce D-branes. Indeed, the original Type I theory can be thought of as having sixteen D9 branes, corresponding to the Neumann boundary conditions in all spatial directions¹⁰.

D-branes are also present in the Type 0 theories. Just like in the IIA (IIB) case, the critical 0A (0B) projection has only even (odd) dimensional branes. However, in each of the Type 0 theories there are two R-R sectors, as opposed to one such sector in their Type II counterparts. This means that the spectrum of D-branes in a critical Type 0 theory is double that of the corresponding Type II model [2]. Unlike the closed Type 0 strings discussed earlier, the open string sectors of the theories *do*

⁹The ends of the string were free to move anywhere in spacetime.

¹⁰~~Sixteen D9-branes are needed to give tadpole cancellation. This leads to the theory having the~~
gauge group $SO(32)$.

contain spacetime fermions.

D-branes are very important objects in the critical string theories examined above. We will see that they also play an important role, in the guise of ZZ and FZZT branes, within the noncritical string theories that we will study in the next section.

1.8 Noncritical String Theory

As was explained earlier, in noncritical string theory we consider the Weyl scaling of the metric as a field in its own right, thus eliminating the problem of the Weyl anomaly. We change the definition of the partition function in (1.25) to:

$$Z = \int \frac{\mathcal{D}X \mathcal{D}g}{V_{\text{diff}}} \exp \left(-S_M[X, g] - \frac{\mu_0}{8\pi} \int d^2\sigma \sqrt{g} \right) \quad (1.44)$$

Here we have introduced a term involving μ_0 , which is the bare *bulk cosmological constant*. This term simply weights surfaces according to their area. Notice that in this definition we simply divide out by the volume of the diffeomorphism orbits. We ignore the volume of the Weyl orbits because the Liouville mode, φ , is now itself a physical field. Some calculation allows one to re-write this partition function as [3]:

$$Z = \int [d\tau] \mathcal{D}_{\hat{g}}\varphi \mathcal{D}_{\hat{g}}[ghosts] \mathcal{D}_{\hat{g}}X e^{-S_M[X, \hat{g}] - S_g[b, c, \bar{b}, \bar{c}, \hat{g}] - S_L[\varphi, \hat{g}]} \quad (1.45)$$

where $[d\tau]$ represents the integration measure over all the appropriate moduli of the surface in question (see Section A.3). S_L is the *Liouville action* given by:

$$S_L = \frac{1}{8\pi} \int d^2\sigma \sqrt{\hat{g}} \left(\hat{g}^{ab} \partial_a \varphi \partial_b \varphi + Q \hat{R} \varphi - \mu e^{2b\varphi} \right), \quad (1.46)$$

$$Q \equiv \sqrt{\frac{25-c}{6}} = \frac{1}{b} + b \quad \Rightarrow \quad b = \frac{1}{\sqrt{24}} (\sqrt{25-c} - \sqrt{1-c}), \quad (1.47)$$

where μ is a renormalised bulk cosmological constant into which μ_0 has been absorbed, and $c \equiv c_{\text{matter}}$ is the central charge of the conformal field theory. With the above prescription one finds that the total central charge is given by:

$$c_{\text{matter}} + c_{\text{Liouv}} + c_{\text{ghost}} = c + (26 - c) - 26 = 0, \quad (1.48)$$

which means that the fake Weyl invariance is never anomalously broken for any value of c we care to choose. This noncritical string theory in dimension $d = c$ can then be re-interpreted as a critical string theory in dimension $D = c + 1$.¹¹

¹¹Strictly this only correct for unitary models, which have $c \geq 0$ [4, 5].

Note that Q and b are only both real for $c \leq 1$. This is therefore the region in which noncritical string theory is well defined. For other values of c the theory is hard to understand. From the form of S_L , and comparing to (1.33), we see (ignoring the μ term for now) that φ is just like an extra spacelike string dimension, with the dilaton given by $\Phi(X) = Q\varphi/\sqrt{2}$. This means that the string coupling is proportional to $\exp(Q\varphi/\sqrt{2})$.¹² So this is a very strange background we are dealing with. At $\varphi = -\infty$ strings are free, whilst at $\varphi = \infty$ strings are infinitely strongly coupled. When $c = 1$ ($D = 2$), with X^0 being timelike, the only spacetime string excitation is the tachyon¹³. Actually, in this theory the name ‘tachyon’ is a misnomer, because it is actually massless¹⁴. The final term in the Liouville action is the cosmological constant term. This serves to strongly suppress the partition function at large values of φ . Physically therefore, we have the phenomenon of tachyons propagating from $\varphi = -\infty$ to some finite value of φ . As the coupling gets stronger and stronger they begin to interact more and more, eventually scattering back into the weak coupling regime (this process is depicted in Figure 1.1) whence they came. The typical location of this scattering is $\varphi \sim \varphi_\mu \equiv -\ln|\mu|$. This is known as the *Liouville wall*.

A quantity that will be useful later is the *string susceptibility*, γ_{str} . This is defined via the partition function at fixed area A :

$$Z(A) = \int \mathcal{D}\varphi \mathcal{D}X e^{-S} \delta \left(\int d^2\sigma \sqrt{\hat{g}} e^{2b\varphi} - A \right). \quad (1.49)$$

The string susceptibility is then defined by taking the limit of this as $A \rightarrow \infty$:

$$Z(A) \sim e^{-\mu A} A^{\frac{1}{2}(\gamma_{str}-2)\chi-1}. \quad (1.50)$$

A simple scaling argument (KPZ scaling) determines [6]:

$$\gamma_{str} = 2 - \frac{Q}{b} = \frac{1}{12} \left(c - 1 - \sqrt{(1-c)(25-c)} \right). \quad (1.51)$$

Notice that this quantity is complex for $1 < c < 25$, suggesting that there is a problem with KPZ scaling in this regime. Using the principle of general covariance we know that we can always couple any two-dimensional conformal field theory to gravity in

¹²For $c = 25$ we have $Q = 0$, and hence the Liouville direction can be Wick rotated into a timelike direction to give us back the familiar case of critical string theory

¹³The physical state conditions at $D = 2$ remove all other excitations.

¹⁴The mass-squared of the ‘tachyon’ is proportional to $2 - D$.

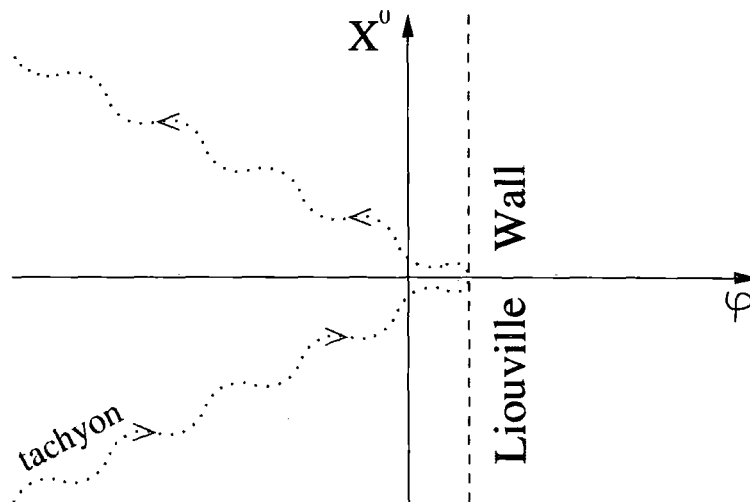


Figure 1.1: A massless tachyon travels from the weak coupling region, $\varphi = -\infty$, to the strong coupling region $\varphi = \infty$. When it reaches a typical distance, known as the *Liouville Wall*, it scatters back to the weak coupling regime.

order to find a $S_M[X, g]$. That is, we can always introduce the worldsheet metric. Because the theory is conformal, we find that the operators of the theory are just the products of the operators of the conformal field theory and those of the Liouville theory (which is just 2D quantum gravity itself). This process is known as the *gravitational dressing* of the operators. Starting with a CFT operator, \mathcal{A}_j^0 , with weights (h_j, \tilde{h}_j) , we couple this to Liouville gravity as $\mathcal{A}_j = e^{2\alpha_j\varphi} \mathcal{A}_j^0$. To be conformally invariant under integration on the worldsheet this must, as we have seen before, have weights $(1, 1)$. This allows us to determine α via the condition:

$$h_j - \alpha_j(\alpha_j - Q) = 1, \quad (1.52)$$

where the conformal weight of the Liouville operator $e^{2\alpha_j\varphi}$ has been calculated via its operator product with $T(z)$. Notice that, for each value of h_j , (1.52) permits two solutions for α_j . It can be shown that the physical solution corresponds to $\alpha_j \leq Q/2$. Those operators dressed with $\alpha_j > Q/2$ do not correspond to local closed-string operators [7], but we will see an interesting interpretation of them later.

One particularly interesting set of models are constructed by coupling Liouville gravity to a special class of conformal field theories having a finite number of primary operators. These conformal field theories are known as *minimal models*, and upon coupling to gravity they will become *minimal string theories*, which are the subject of

Chapter 3. Before studying these models we will explore, in Chapter 2, the integrable systems that will play such an important role in these minimal string theories later on.

Chapter 2

Integrable Systems and The KdV Equation

2.1 What is an Integrable System?

2.1.1 Hamilton's Equations and Action-Angle Variables

Consider a mechanical system with a finite number of degrees of freedom. This system is described in terms of positions, q_i ($i = 1, \dots, n$), and momenta, p_i , and has the usual equations of motion in terms of the Hamiltonian, $H(q_i, p_i)$ [8]:

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}, \quad (2.1)$$

where $\dot{}$ represents differentiation with respect to time and, as usual, $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = 0$ and $\{p_i, p_j\} = 0$, with the canonical Poisson bracket, $\{ , \}$, defined as:

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \quad (2.2)$$

The coordinates (q_i, p_i) of the above Hamiltonian system form a $2n$ -dimensional phase space. The system is defined as *completely integrable* if and only if there exist exactly n functionally independent conserved quantities that are in involution (that is, the Poisson brackets of these conserved quantities with one another vanish). If the conserved quantities are not in involution then the system is merely *integrable*.

If a system is integrable then it means that it is possible, at least in principle, to obtain an exact solution for the motion of the system. This is known as *Liouville's Theorem*, a proof of which exists in many texts on classical mechanics ([9])

for instance). In this respect the words *integrable* and *solvable* are synonymous. The conserved quantities associated with integrability are known as *integrals of motion* $K_i = K_i(q_i, p_i)$. They satisfy:

$$\dot{K}_i = \{K_i, H\} = 0, \quad \{K_i, K_j\} = 0, \quad (2.3)$$

where the second equation expresses the involution condition.

It follows that the time-independent Hamiltonian of the system must be expressible as a linear combination of the K_i . Complete integrability implies that we can make a canonical transformation to *action-angle* variables. To do this we consider the $K_i = K_i(q_i, p_i)$ as the momenta in the new coordinates. We then search for the canonically conjugate variables, $\theta_i = \theta_i(q_i, p_i)$, such that:

$$\{\theta_i, \theta_j\} = 0 = \{K_i, K_j\}, \quad \{\theta_i, K_j\} = \delta_{ij} \quad i, j = 1, 2, \dots, n. \quad (2.4)$$

Now, since $H = H(K_i)$, we can re-write Hamilton's equations (2.1) as:

$$\dot{K}_i = \{K_i, H(K_j)\} = 0, \quad \dot{\theta}_i = \{\theta_i, H(K_j)\} = f_i(K_j), \quad (2.5)$$

for some constants f_i . The first equation above is merely the statement that the K_i 's are conserved: $K_i(t) = K_i(0)$. The second equation implies:

$$\theta_i = f_i t + \alpha_i, \quad (2.6)$$

where the α_i are constants that can be fixed from the initial conditions. So, for a completely integrable system, the time-evolution in action-angle variables is just a linear flow on an n -dimensional torus, parametrised by the angle variables θ_i .

So now the whole problem has been reduced to finding the canonical transformation, $\theta_i = \theta_i(q_j, p_j)$, and then to subsequently invert the solution to obtain the evolution as:

$$q_i = q_i(f_j, \alpha_j, t), \quad p_i = p_i(f_j, \alpha_j, t). \quad (2.7)$$

In practice it is this latter step that causes all the difficulties, since Liouville's Theorem merely promises the existence of solutions, and does not guarantee they will be possible to find.

So, when will a given Hamiltonian, $H(q_i, p_i)$, correspond to a completely integrable system? It turns out that there is no systematic way of determining this. However, if there exists a so-called *Lax pair* then we will show that the system is indeed completely integrable. A Lax pair is a pair of $N \times N$ matrix-valued functions, $P(q, p)$ and $Q(q, p)$, that depend upon the dynamical variables in such way that the *Lax equation*

$$\dot{Q} = [Q, P], \quad (2.8)$$

is equivalent to Hamilton's equations (2.1). The dimension N of these Lax matrices is not *a priori* known, and the Lax pair is not unique because we are always free to perform a gauge transformation of the form:

$$Q \rightarrow G^{-1}QG \quad P \rightarrow G^{-1}PG - G^{-1}\dot{G}. \quad (2.9)$$

It is now trivial to form the integrals of motion K_i from the Lax pair. Simply take the trace of powers of Q :

$$K_i \equiv \text{Tr } Q^i \quad \Rightarrow \quad \dot{K}_i = i \text{Tr } (Q^{i-1}[Q, P]) = 0, \quad (2.10)$$

which follows from trivial trace manipulations; here $i \in \mathbb{Z}^+$. Note that only N of the K_i will be functionally independent because of Cayley's theorem [8]. So, providing that $n \leq N$, we have enough constants of motion for complete integrability.

More generally one can have Lax Pairs that depend on an extra parameter, $\tilde{\lambda}$, that has no relation to the physics of the system: each value of $\tilde{\lambda}$ merely gives a different Lax Pair. $\tilde{\lambda}$ is usually referred to as the *spectral parameter*. The Lax pair will prove to be very important in the context of the KdV equation and minimal string theory, and so will be returned to later.

For certain constrained problems it proves useful to work in noncanonical coordinates, which have a more complex Poisson bracket structure. This forms quite an elegant structure and will be reviewed in Section B.1.

2.2 The Korteweg-de Vries Equation

We can extend the above results to the case of a system with an infinite number of degrees of freedom. That is, starting from coordinates $u_i(t)$, we can take the continuum limit to obtain $u(x, t)$, where x is a continuous variable. The function

$u(x, t)$ will then satisfy a partial differential equation in x and t . It follows that such a system has an infinite number of conserved quantities; hence it is infinitely constrained. Such a system can give rise to *solitonic* solutions. A soliton is localised and retains its shape, even after collisions. For this to happen it is intuitively obvious that there needs to be an infinite number of conservation laws. The canonical example of a soliton is a tsunami or tidal wave: a water wave that can travel huge distances without dissipating. The first recorded instance of such a *solitary wave* was observed by J. Scott Russell on the Edinburgh Union Canal in 1834 [10]. The *Korteweg-de Vries* (KdV) equation was originally formulated by Boussinesq in 1877 to describe such waves propagating in a rectangular channel [11]. It was later rediscovered by Korteweg and de Vries and is given by:

$$\alpha \frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}, \quad (2.11)$$

which is a non-linear equation in the function $u(x, t)$. Here α is a constant that can be absorbed into a rescaling of t . We will set $\alpha = -4$ to match with later conventions. One can think of x as the co-moving¹ position coordinate along the rectangular channel, and of u as the net displacement of the water from its mean height.

For future use it will be instructive to briefly examine some of the many properties of the KdV equation. Again, this will follow the treatment presented in [12], and a far more detailed discussion can be found there. First let us consider the symmetries of the KdV equation:

Translation Invariance. The KdV equation is invariant under the transformation

$$t \mapsto t + t_0, \quad x \mapsto x + x_0, \quad \text{where } t_0 \text{ and } x_0 \text{ are constants.}$$

Rescaling Invariance. A rescaling of the form $x \mapsto cx$, $t \mapsto c^3 t$, $u \mapsto c^{-2} u$, where c is a non-zero constant, is another symmetry of KdV. We can use this to assign a *scaling dimension* to x , t and u . We denote the scaling dimension of a quantity f as $[f]$; and write $[t] = 3[x]$, $[u] = -2[x]$, in accordance with the relative powers of c in the rescaling invariance.

Galilean Invariance. The transformation $x \mapsto x - \frac{3}{2} \tilde{\lambda} t$, $t \mapsto t$, $u \mapsto u + \tilde{\lambda}$, also leaves the KdV equation invariant for all values of $\tilde{\lambda}$. This invariance will turn

¹That is, after boosting to a frame in which all the waves appear to move to the left.

out to be most useful in the context of minimal string theory.

Another point of note is that there is no symmetry under $x \mapsto -x$, which is a consequence of being in the co-moving frame.

2.2.1 Hamilton's Equations for KdV

To proceed we define the continuum version of the Poisson bracket:

$$\{F[u], G[u]\} = \int dx dy \frac{\delta F}{\delta u(x)} \{u(x), u(y)\} \frac{\delta G}{\delta u(y)}, \quad (2.12)$$

where $F[u]$ and $G[u]$ are now *functionals* and $\delta/\delta u(x)$ is the standard functional derivative:

$$\frac{\delta F[u(x)]}{\delta u(y)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[u(x) + \epsilon \delta(x - y)] - F[u(x)]), \quad (2.13)$$

where $\delta(x - y)$ is the Dirac delta function. Using this definition it is trivial to prove that the following choice of Hamiltonian and fundamental Poisson bracket leads directly to the KdV equation:

$$H[u] = \frac{1}{2} \int_{-\infty}^{\infty} dx \left(u(x)^3 - \left(\frac{\partial u(x)}{\partial x} \right)^2 \right). \quad (2.14)$$

$$\{u(x), u(y)\} = \frac{\partial}{\partial x} \delta(x - y). \quad (2.15)$$

The Poisson bracket is obviously antisymmetric, and a little work yields both its inverse and the Jacobi identity². A curiosity of the KdV equation is that there exists a second possible Hamiltonian and Poisson Bracket structure:

$$\hat{H}[u] = -\frac{1}{2} \int_{-\infty}^{\infty} dx u^2(x), \quad (2.16)$$

$$[u(x), u(y)] = \left(\frac{\partial^3}{\partial x^3} - 4u(x) \frac{\partial}{\partial x} - 2 \frac{\partial u(x)}{\partial x} \right) \delta(x - y), \quad (2.17)$$

where $[,]$ denotes the new Poisson bracket. A proof of (2.16) along the lines of that which led to (2.14) can be found in [12]. This peculiarity will prove crucial in proving that the KdV equation is integrable.

²The Jacobi identity is explained in Section B.1.

2.2.2 Travelling Wave Solutions and Solitons

One of the properties of the KdV equation is that for a given initial condition, $u(x, 0)$, there exists only one solution to the KdV equation. That is, solutions are unique. Here we consider a travelling wave solution of KdV, $u(x, t) = u(x + ct)$, with speed c . Inserting this into (2.11) we obtain:

$$-4c \frac{\partial u}{\partial x} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}. \quad (2.18)$$

At $t = 0$ we can integrate to obtain:

$$-\frac{d^2 u}{dx^2} + 3u^2 + 4cu = \text{constant}. \quad (2.19)$$

If we assume that $u(x, t)$ vanishes as $x \rightarrow \pm\infty$ then the constant term in (2.19) must also vanish. Multiplying (2.19) by du/dx and integrating again allows one to obtain the closed form solution (with the boundary condition $du/dx = 0$ at $x = 0$):

$$u(x, t) = -2c \operatorname{sech}^2 [\sqrt{c} (x + ct)]. \quad (2.20)$$

Figure 2.1 shows a graph of this solution with $c = 1$ at $t = 0$.³ Notice that this highly localised solution only travels to the left (positive c); and if we try to make c negative then we instead get an oscillatory solution. Another generic feature of soliton solutions is that they maintain their shape as they move. Also, the taller the wave is, the faster it travels. This is clearly evident from the form of (2.20).

2.3 More on KdV

Many of the important qualities of the KdV equation rely upon its integrability. It is therefore important to establish that this property does indeed hold. This is done in detail in Section B.2. The proof will introduce some important concepts that we will review here.

In the context of KdV, a conserved quantity will be a functional $\mathcal{H}_i[u]$, satisfying the relation:

$$\frac{d\mathcal{H}_i[u]}{dt} = \{\mathcal{H}_i[u], H\} = 0. \quad (2.21)$$

³In the above discussion $u(x)$ is defined with a relative minus sign from the standard literature. This will prove useful later. However, to make the graph look more 'wave-like', $-u(x)$ is plotted on the vertical axis in Fig.2.1.

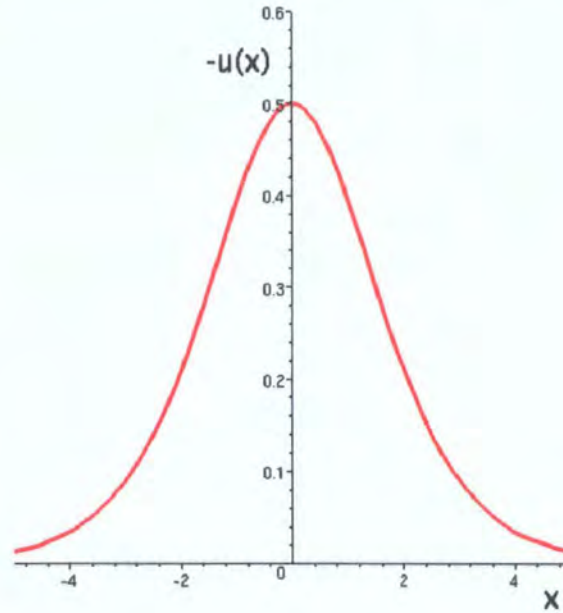


Figure 2.1: A 1-soliton solution of the KdV equation.

We have already seen that there should be an infinite number of $\mathcal{H}_i[u]$, obeying $\{\mathcal{H}_i, \mathcal{H}_j\} = 0$. The first three are:

$$\mathcal{H}_0[u] = \int_{-\infty}^{\infty} dx u(x, t), \quad \mathcal{H}_1[u] = \hat{H}, \quad \mathcal{H}_2[u] = H. \quad (2.22)$$

In general they satisfy the following recursion relation (with $\mathcal{H}_{-1} = 0$):

$$(\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_{i-1}}{\delta u(x)} = \partial \frac{\delta \mathcal{H}_i}{\delta u(x)} \quad i = 0, 1, 2, \dots, \quad (2.23)$$

where we have defined $\partial \equiv \partial/\partial x$ and $u_x \equiv \partial u/\partial x$ for clarity.⁴ It turns out that the \mathcal{H}_i allow us to define a whole hierarchy of KdV equations.

2.3.1 A Hierarchy of KdV Equations

Recall that the KdV Equation can be written as:

$$\begin{aligned} \alpha u_t &= [u, \mathcal{H}_1] = (\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_1}{\delta u(x)} \\ &= \{u, \mathcal{H}_2\} = \partial \frac{\delta \mathcal{H}_2}{\delta u(x)} = 6uu_x - u_{xxx}, \end{aligned} \quad (2.24)$$

⁴Later on when we differentiate quantities with subscripts, such as S_i for example, we will write its differential as $S_{i,x}$ to avoid confusion.

where $\mathcal{H}_1 \equiv \hat{H}$ and $\mathcal{H}_2 \equiv H$. Let us study the equations associated with using an arbitrary member of the \mathcal{H}_i as the fundamental Hamiltonian. Since all of the \mathcal{H}_i are in involution we know from the start that this new Hamiltonian will also be an integrable system with the same integrals of motion. So we can write:

$$\begin{aligned} \alpha_n u_{t_n} &= [u, \mathcal{H}_n] = (\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_n}{\delta u(x)}, \\ &= \{u, \mathcal{H}_{n+1}\} = \partial \frac{\delta \mathcal{H}_{n+1}}{\delta u(x)} \quad n = 0, 1, 2, \dots, \end{aligned} \quad (2.25)$$

where t_n is the time variable associated with each equation⁵. This full set of equations is known as the *KdV hierarchy*. We can therefore think of u as a function $u(x, t_0, t_1, t_2, \dots)$ of all the t_n ; and, since the \mathcal{H}_i are in involution, flows in the different time variables will commute. The first three members of the KdV hierarchy are given below:

$$\begin{aligned} \alpha_0 u_{t_0} &= -u_x, & \alpha_1 u_{t_1} &= 6uu_x - u_{xxx}, \\ \alpha_2 u_{t_2} &= 20u_x u_{xx} + 10uu_{xxx} - u_{xxxx} - 30u^2 u_x. \end{aligned} \quad (2.26)$$

The first of these equations involving t_0 represents a chiral wave moving only to the right. Now that we have studied the KdV equation in great detail, it is time to introduce a closely related system known as *modified KdV* (mKdV). We will do this by studying a transformation that connects the two: *the Miura map*.

2.3.2 The Miura Map

Consider the effect of the following Riccati transformation on (2.11):

$$u(x, t) = v(x, t)^2 - v_x(x, t). \quad (2.27)$$

Substitution of (2.27) into (2.11) eventually leads to the following equation:

$$4(\partial - 2v)v_t = (\partial - 2v)(v_{xxx} - 6v^2 v_x). \quad (2.28)$$

A consequence of this is that any solution of:

$$4v_t = v_{xxx} - 6v^2 v_x, \quad (2.29)$$

⁵ t_1 is what we were previously calling t

is automatically a solution of the KdV equation (2.11); though analysis of (2.28) confirms that the converse does not necessarily follow. The transformation (2.27) is known as the Miura map, and the equation it yields (2.29) is the famous mKdV equation.

We can now use the Galilean invariance (see Section 2.2) of the KdV equation to ‘inject’ a parameter into the Miura map. All we have to do is apply the same Galilean transformation to both the Miura map (2.27), and the mKdV equation (2.29). This gives:

$$u = v^2 - v_x - \tilde{\lambda}, \quad (2.30)$$

$$4v_t = v_{xxx} - 6(v^2 - \tilde{\lambda})v_x, \quad (2.31)$$

using the same notation as in Section 2.2. Indeed, this result can be verified by direct substitution of this new Miura map into (2.11). The method of using Galilean invariance to generalise transformations of this kind has been studied, albeit in the guise of the geometry of pseudospherical surfaces, as far back as 1882 by Bäcklund. In fact, transformations of this kind, between one differential equation and another, are often referred to as *Bäcklund transformations* for this reason. Bäcklund transformations will be discussed in greater detail later, but for now let us note that the original, parameter free, Miura map (2.27) is referred to in the literature as a *Bianchi transformation*, \mathbb{B}_0 . In 1883, Lie showed that a full Bäcklund transformation $\mathbb{B}_{\tilde{\lambda}}$, such as (2.30), can be thought of as a conjugation of Lie transformations $\mathbb{L}_{\tilde{\lambda}}$, $\mathbb{L}_{\tilde{\lambda}}^{-1}$, which in this case describe the Galilean invariance, with \mathbb{B}_0 [11]. Specifically we have $\mathbb{B}_{\tilde{\lambda}} = \mathbb{L}_{\tilde{\lambda}}^{-1} \mathbb{B}_0 \mathbb{L}_{\tilde{\lambda}}$.

2.4 A Schrödinger Problem

As was explained in subsection (2.3.2), the Miura map (2.30) will generate a solution of the KdV equation (2.11) from any given solution of the mKdV equation (2.31). However, the converse is not necessarily true because in general the Miura map is not invertible. However, it is clear from Section B.2 that since the KdV equation is integrable then so is the mKdV equation, and with the same integrals of motion. So it seems plausible that we can find a relatively simple way to obtain $v(x, t)$ from a

given $u(x, t)$. The way to do this is to linearise the system by writing:

$$\psi(x, t) = e^{-\int dx v(x, t)} \Rightarrow v = -\frac{\psi_x}{\psi}, \quad (2.32)$$

which allows us to write the Miura transformation (2.30) as:

$$Q\psi \equiv (\partial^2 - u)\psi = \tilde{\lambda}\psi, \quad (2.33)$$

which is the time-independent Schrödinger equation with potential $u(x, t)$ and energy eigenvalue $-\tilde{\lambda}$. Therefore, assuming $u(x, t)$ is smooth and vanishes at $x \rightarrow \pm\infty$, we see that $\tilde{\lambda} > 0$ will correspond to bound-state wavefunctions, whereas $\tilde{\lambda} < 0$ will correspond to travelling-wave solutions. Given a potential, $u(x, t)$, we can find the associated solution of the mKdV equation by solving Schrödinger's equation for $\psi(x, t)$, and then substituting the result into (2.32). In the literature, $\psi(x, t)$ is known as the *Baker function*.

Notice that the eigenvalue $\tilde{\lambda}$ in (2.33) is independent of time, because it enters as the result of the Galilean invariance of the KdV equation. This poses the question of whether or not there is a linear operator, with time-independent eigenvalues, analogous to Q (in (2.33)) for other nonlinear integrable systems. To investigate this we recall the standard quantum mechanical interpretation of Q in the *Heisenberg picture*, wherein the operators, rather than the states, evolve in time. We have:

$$Q(u(x, t)) \equiv Q(t) = U(t)Q(0)U^\dagger(t), \quad (2.34)$$

where $U(t)$ is a unitary time evolution operator. Solving (2.34) for $Q(0)$ and differentiating with respect to t , we find:

$$\frac{\partial U^\dagger(t)}{\partial t} Q(t) U(t) + U^\dagger(t) \frac{\partial Q(t)}{\partial t} U(t) + U^\dagger(t) Q(t) \frac{\partial U(t)}{\partial t} = 0, \quad (2.35)$$

along with:

$$U^\dagger(t)U(t) = 1 \Rightarrow \frac{\partial U^\dagger(t)}{\partial t} U(t) + U^\dagger(t) \frac{\partial U(t)}{\partial t} = 0. \quad (2.36)$$

One way to satisfy (2.36) would be to set:

$$\frac{\partial U(t)}{\partial t} = P(t)U(t), \quad (2.37)$$

where $P(t)$ is anti-Hermitian. Using (2.37) we can rewrite (2.35) as:

$$\begin{aligned} U^\dagger(t) \left(\frac{\partial Q(t)}{\partial t} - [P(t), Q(t)] \right) U(t) &= 0, \\ \Rightarrow \frac{\partial Q(t)}{\partial t} &= [P(t), Q(t)]. \end{aligned} \quad (2.38)$$

So, if we can find an anti-hermitian operator, $P(t)$, that is multiplicative and proportional to the time evolution of $u(x, t)$ (since $Q(t)$ is only linearly dependent on $u(x, t)$), then we know for sure that the eigenvalues of the equation $Q\psi = \lambda\psi$ will be independent of t . This evolution of $Q(t)$ is known as an *isospectral deformation*.

We can also evolve ψ in time using the same operators:

$$\psi(t) = U(t)\psi(0) \quad \Rightarrow \quad \frac{\partial\psi(t)}{\partial t} = \frac{\partial U(t)}{\partial t}\psi(0) = P(t)\psi(t). \quad (2.39)$$

Finally in this section, let us compare (2.38) with (2.8). It seems that the operators $Q(t)$ and $P(t)$ form the analogue of the Lax pair for continuous systems like KdV. This is indeed the case; and in the next subsection we will see an elegant structure that allows us to derive a whole family of Lax pairs corresponding to integrable systems.

2.4.1 Pseudodifferential Operators and Lax Pairs

A *pseudodifferential operator* is a natural generalisation of the operator ∂ to encompass negative and even fractional powers of that basic differential⁶. These operators have many elegant properties that will be used below. For our purposes we will state many of these properties without proof. The interested reader can refer to reference [10], which delights in expounding these properties in rigorous detail.

We start by defining a differential operator Q_n , which is a generalisation of the Q operator in (2.33):

$$Q_n = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0, \quad (2.40)$$

where the $n - 1$ functions $u_i(x, t)$ are generalisations of $u(x, t)$ in (2.33).⁷ We next introduce a pseudodifferential operator X defined by:

$$X = \sum_{i=-\infty}^m X_i \partial^i, \quad m \in \mathbb{Z}, \quad (2.41)$$

⁶In general, an operator with both fractional and negative powers of differentials is known as *pseudodifferential*; whereas an operator with just negative powers is merely *microdifferential*. From now on we will refer to all such operators as pseudodifferential operators.

⁷Equation (2.40) could also contain a $u_{n-1}\partial^{n-1}$ term, but this can always be removed by a simple transformation.

for some X_m ; here m is an arbitrary integer known as the *order* of the pseudodifferential operator. It is clear that X contains negative powers of ∂ . Naively these correspond to integrations as opposed to differentiations. Here we define ∂^{-1} via the following commutation rule:

$$\partial^{-1} f = \sum_k (-1)^k f^{(k)} \partial^{-1-k}. \quad (2.42)$$

We further define:

$$X_+ = \sum_{i \geq 0} X_i \partial^i, \quad X_- = \sum_{i < 0} X_i \partial^i, \quad \text{res}(X) = X_{-1}. \quad (2.43)$$

Let us study the following operator:

$$P_m = (Q^{m/n})_+ \equiv Q_+^{m/n}. \quad (2.44)$$

Here we calculate $Q^{m/n}$ by defining it to be a pseudodifferential operator of order m :

$$P_m = \sum_{i=-\infty}^m P_{m,i} \partial^i, \quad m \in \mathbb{Z}^+, \quad (2.45)$$

and then calculating the coefficients $P_{m,i}$ via the relationship $(Q^{1/n})^n = Q$. An example will be given below. For now let us calculate $[P_m, Q]$:

$$[P_m, Q] = [Q^{m/n} - Q_-^{m/n}, Q] = -[Q_-^{m/n}, Q], \quad (2.46)$$

where we have used the result that Q commutes with any other power of Q . Notice that the operator on the left-hand side of (2.45) is purely differential (i.e. $[P_m, Q]_- = 0$), whereas the right-hand side is a product of two operators of order -1 and n , meaning that the resulting operator is of order less than or equal to $n-2$.⁸ Comparing the nature of $[P_m, Q]$ to the definition of (2.40) we see that it is sensible to write the following differential equation:

$$\partial_{t_m} Q = [P_m, Q], \quad (2.47)$$

which is actually a set of $n-2$ equations for the u_i . This is a Lax equation defining an integrable system, the conserved quantities of which will be derived below. We see from (2.39) that this Lax equation (2.47) is equivalent to the equations:

$$Q\psi = \tilde{\lambda}\psi, \quad P_m\psi = \partial_{t_m}\psi. \quad (2.48)$$

⁸The nature of the operators means that the possible ∂^{n-1} term of the right-hand commutator in (2.46) is always zero.

Let us elucidate the above procedure with some examples:

- Choosing $(n, m) = (2, 3)$ and $Q = \partial^2 - u$, we calculate P_3 by evaluating $QQ^{1/2}$.

This gives:

$$P_3 = \partial^3 - \frac{3}{2}u\partial - \frac{3}{4}u_x, \quad (2.49)$$

which yields the Lax equation:

$$-4u_{t_3} = 6uu_x - u_{xxx}, \quad (2.50)$$

which is once again the familiar KdV equation.

- Choosing $(n, m) = (3, 2)$ and $Q = \partial^3 - u\partial - v$, we find that P_2 is given by:

$$P_2 = \partial^2 - \frac{2}{3}u, \quad (2.51)$$

which yields the Lax equations:

$$u_{t_2} = -u_{xx} + 2v_x, \quad v_{t_2} = v_{xx} - \frac{2}{3}u_{xxx} + \frac{2}{3}uu_x. \quad (2.52)$$

Eliminating v we obtain:

$$u_{t_2 t_2} = -\frac{1}{3}u_{xxxx} + \frac{2}{3}(uu_x)_x, \quad (2.53)$$

which is known as the *Boussinesq* equation.

The hierarchy of integrable systems with $n = 2$ and $Q = \partial^2 - u$ is the same KdV hierarchy of equations that we studied in (2.26) (up to rescalings). Each choice of n defines its own KdV-like hierarchy such as the $n = 3$ Boussinesq hierarchy.

We can now proceed to find a general formula for the conserved quantities of the hierarchies defined above. If the system is indeed integrable then there should be an infinite number of these. To find them we need to employ two theorems from the theory of pseudodifferential operators (both of which are proved in [10]):

Theorem 1 If $\partial_{t_m} Q = [P_m, Q]$ then $\partial_{t_m} Q^{m/n} = [P_m, Q^{m/n}]$

Theorem 2 If X and Y are pseudodifferential operators then $\text{res}([X, Y])$ is a total derivative.

The integrals of motion of the Lax equation (2.47) are then given by:

$$J_k = \int \text{res}(Q^{k/n}) dx, \quad k = 1, 2, \dots \quad (2.54)$$

To see this we write:

$$\partial_{t_m} J_k = \int \text{res}(\partial_{t_m} Q^{k/n}) dx = \int \text{res}([P_m, Q^{k/n}]) dx = 0, \quad (2.55)$$

using the two theorems above. Note that if k is a multiple of n then the first integral degenerates, $J_k = 0$.

We can actually generalise the above procedures even more, by allowing our basic operator, (2.40), to include matrix-valued functions U_i . This will generate the Zakharov-Shabat integrable hierarchy, which is the subject of Section B.3. It is this hierarchy that will turn out to underpin certain Type 0B minimal string theories.

2.5 Multi-Soliton Solutions and Bäcklund Transformations

The Schrödinger problem underpinning the KdV hierarchy can tell us a lot about the solutions of the hierarchy themselves. It turns out that the bound states ($\tilde{\lambda} > 0$) of the Baker function, $\psi(x, t)$, correspond to soliton solutions of the KdV hierarchy. This is explored in Section B.4. This forms the basis for the elegant technology of *inverse scattering*, wherein solutions of the KdV hierarchy, $u(x, t)$, are constructed from knowledge of the bound and travelling-wave states of $u(x, 0)$. This is the subject of Section B.5.

In the present section we will derive the Bäcklund transformation that interrelates solutions of the KdV hierarchy. This will be explored from the point of view of $u(x, t)$ itself, rather than that of the Baker function, $\psi(x, t)$. In terms of the Baker function, the existence of this Bäcklund transformation is underpinned by a supersymmetric quantum mechanical structure. This will be explored in Section B.6, but will reappear in Chapter 5 in the context of minimal Type 0A strings.

Recall the Miura map (2.30) and the mKdV equation (2.31). In subsection 2.3.2 we discussed how, given a solution of the mKdV equation, we could use the Miura map to obtain a solution of the KdV equation (2.50). Notice that if v is a solution of

(2.31) then $-v$ is also a solution. Denoting the solution of KdV associated to v as u_- and the solution of KdV associated to $-v$ as u_+ we can write:

$$\begin{aligned} u_- &= v^2 - v_x - \tilde{\lambda}, \\ u_+ &= v^2 + v_x - \tilde{\lambda}. \end{aligned} \quad (2.56)$$

Addition and subtraction of these two equations leads to:

$$\begin{aligned} u_+ + u_- &= 2v^2 - 2\tilde{\lambda}, \\ u_+ - u_- &= 2v_x. \end{aligned} \quad (2.57)$$

Writing $(w_-)_x \equiv u_-$ and $(w_+)_x \equiv u_+$, we can integrate the latter equation⁹ and substitute into the first to obtain:

$$(w_+ + w_-)_x = \frac{1}{2}(w_+ - w_-)^2 - 2\tilde{\lambda}. \quad (2.58)$$

This is the spatial part of the *auto-Bäcklund transformation* of the KdV equation. As was explained in subsection 2.3.2, a Bäcklund transformation is the name given to a transformation that relates solutions of one equation to solutions of another. If it relates solutions of an equation to other solutions of the same equation (as in (2.58)) then it is an auto-Bäcklund transformation. It is easy to show using (2.50) that $w(x, t)$ satisfies the potential KdV (pKdV) equation:

$$-4w_t = 3w_x^2 - w_{xxx}. \quad (2.59)$$

Given one solution w_- of pKdV, we can use (2.58) to obtain a second solution w_+ . In addition, because $\tilde{\lambda}$ is arbitrary, the new solution will actually be a one parameter family of solutions. We can use the Miura map (2.30) to eliminate all spatial derivatives of $v(x, t)$ from the mKdV equation (2.31). This yields:

$$\begin{aligned} -4v_t &= 2v^2 u_- - 4\tilde{\lambda}v^2 - 2u_-^2 + 2\tilde{\lambda}u_- + 4\tilde{\lambda}^2 + 2v(u_-)_x + (u_-)_{xx}, \\ 4v_t &= 2v^2 u_+ - 4\tilde{\lambda}v^2 - 2u_+^2 + 2\tilde{\lambda}u_+ + 4\tilde{\lambda}^2 - 2v(u_+)_x + (u_+)_{xx}. \end{aligned} \quad (2.60)$$

Subtracting the second of these equations from the first and using (2.57) and (2.58) leads to:

$$-4(w_+ - w_-)_t = 3(w_+ - w_-)_x(w_+ + w_-)_x - (w_+ - w_-)_{xxx}. \quad (2.61)$$

⁹Setting integration constants to zero.

This equation is the time part of the Bäcklund transformation.

We can use the Bäcklund transformation (2.58) to generate soliton solutions of the KdV hierarchy. To do this we start with the trivial solution of pKdV, namely $w_- = 0$. Substitution into (2.61) just recovers the pKdV equation we started with. Substitution into (2.58) yields:

$$(w_+)_x = \frac{1}{2}w_+^2 - 2\tilde{\lambda}. \quad (2.62)$$

We can integrate this up to obtain the solution:

$$w_+(x, t) = \begin{cases} -2/(x + c(t)) & \tilde{\lambda} = 0, \\ -2\sqrt{\tilde{\lambda}} \tanh(x\sqrt{\tilde{\lambda}} + c(t)) & \tilde{\lambda} \neq 0, \end{cases} \quad (2.63)$$

where $c(t)$ is a time dependent integration constant. Differentiation yields $u(x, t)$ given by:

$$u(x, t) = \begin{cases} 2/(x + c(t))^2 & \tilde{\lambda} = 0, \\ -2\tilde{\lambda} \operatorname{sech}^2(x\sqrt{\tilde{\lambda}} + c(t)) & \tilde{\lambda} \neq 0. \end{cases} \quad (2.64)$$

For $\tilde{\lambda} = 0$, substitution of $u(x, t)$ into the KdV equation implies that $c(t)$ must be independent of time in this case. For $\tilde{\lambda}$ non-zero we find that $c(t) = \tilde{\lambda}^{3/2}t$, which gives:

$$u(x, t) = \begin{cases} 2/(x + c)^2 & \tilde{\lambda} = 0, \\ -2\tilde{\lambda} \operatorname{sech}^2(\sqrt{\tilde{\lambda}}(x + \tilde{\lambda}t)) & \tilde{\lambda} \neq 0, \end{cases} \quad (2.65)$$

which (for $\tilde{\lambda} \neq 0$) is just the 1-soliton solution of (2.20) with $c = \tilde{\lambda}$. In fact, it turns out that successive applications of the Bäcklund transformation serve to alter the soliton number of the solution by unity [12]. So we can think of it as creating and destroying solitons. We can plug the 1-soliton solution back into the Bäcklund transformation to generate the 2-soliton solution. Solving for this is complicated however. Instead, we will use what is known as the *permutability theorem*.

2.5.1 The Permutability Theorem

It can be shown that Bäcklund transformations satisfy the theorem of permutability. Consider a solution of pKdV, w . We can apply the Bäcklund transformation to this solution, choosing $\tilde{\lambda} = \tilde{\lambda}_1$, to obtain an new solution of pKdV, w_1 . We can repeat this

procedure with $\tilde{\lambda} = \tilde{\lambda}_2$ to obtain a solution w_{12} . Similarly, we can apply the Bäcklund transformation with $\tilde{\lambda} = \tilde{\lambda}_2$ directly to w to obtain a solution w_2 ; and then Bäcklund transform again with $\tilde{\lambda} = \tilde{\lambda}_1$ to obtain w_{21} . Writing the Bäcklund transformation as $\mathbb{B}_{\tilde{\lambda}}$, we have:

$$w_1 = \mathbb{B}_{\tilde{\lambda}_1}(w), \quad w_2 = \mathbb{B}_{\tilde{\lambda}_2}(w), \quad w_{12} = \mathbb{B}_{\tilde{\lambda}_2}(w_1), \quad w_{21} = \mathbb{B}_{\tilde{\lambda}_1}(w_2). \quad (2.66)$$

The theorem of permutability states that the order in which Bäcklund transformations are performed is irrelevant. So $w_{12} = w_{21}$; or, alternatively, $[\mathbb{B}_{\tilde{\lambda}_1}, \mathbb{B}_{\tilde{\lambda}_2}] = 0$. Putting this information into the Bäcklund transformation itself, we obtain:

$$\begin{aligned} (w_1 + w)_x &= \frac{1}{2}(w_1 - w)^2 - 2\tilde{\lambda}_1, & (w_{12} + w_1)_x &= \frac{1}{2}(w_{12} - w_1)^2 - 2\tilde{\lambda}_2, \\ (w_2 + w)_x &= \frac{1}{2}(w_2 - w)^2 - 2\tilde{\lambda}_2, & (w_{12} + w_2)_x &= \frac{1}{2}(w_{12} - w_2)^2 - 2\tilde{\lambda}_1. \end{aligned} \quad (2.67)$$

Eliminating the derivative terms we obtain:

$$\begin{aligned} \frac{1}{2} [(w_{12} - w_2)^2 + (w_1 - w)^2 - (w_{12} - w_1)^2 - (w_2 - w)^2] &= 4(\tilde{\lambda}_2 - \tilde{\lambda}_1), \\ \Rightarrow (w - w_{12})(w_2 - w_1) &= 4(\tilde{\lambda}_2 - \tilde{\lambda}_1), \end{aligned} \quad (2.68)$$

which gives the final result:

$$w_{12} = w - 4 \frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{w_2 - w_1}, \quad (2.69)$$

which gives a completely algebraic method to compute the solutions generated by repeated applications of the Bäcklund transformation.

Chapter 3

Minimal String Theory and Matrix Models

In Chapter 1 we studied the physics of conformal field theories. By coupling these theories to worldsheet gravity we were led to string theory. In Chapter 2 we apparently went off at a tangent and undertook a detailed run-through of integrable models. In this chapter we will begin to see how these apparently disparate aspects of mathematical physics come together in the context of *minimal string theory*. In the process we will introduce the concept of the *matrix model*, which is an invaluable calculational tool linking minimal string theory and integrable models. We start this chapter with some more conformal field theory:

3.1 Minimal Conformal Field Theories

Recall the discussion in Section 1.3 on conformal families in CFTs. Here we will discuss what happens if such a family is *degenerate*, and this will lead us to the concept of a minimal model [13]. A primary operator, \mathcal{O}_j , with weights $(\Delta_j, \tilde{\Delta}_j)$, is associated to a conformal family with secondary operators at integer spaced weights $(\Delta_j + k, \tilde{\Delta}_j + \tilde{k})$. Also recall that the correlators involving these secondary operators can always be calculated in terms of correlators involving the corresponding primary operators, and that the associated primary states satisfy $L_n|j\rangle = 0$ if $n > 0$ and $L_0|j\rangle = \Delta_j|j\rangle$. A conformal family, $[\mathcal{O}_j]$, is said to be degenerate if it contains a *null-vector* $|\chi\rangle$ satisfying:

$$L_n|\chi\rangle = 0 \quad \text{for } n > 0 \quad L_0|\chi\rangle = (\Delta_j + K)|\chi\rangle, \quad (3.1)$$

for some integer K . It turns out that this only happens for special values of $\Delta_j = \Delta_{(r,s)}$, parametrised by two positive non-zero integers (r, s) and given by [14]:

$$\Delta_{(r,s)} = \Delta_0 + \left(\frac{\alpha_+}{2}r + \frac{\alpha_-}{2}s \right)^2, \quad \text{with } \Delta_0 = \frac{c-1}{24} \quad \text{and} \quad \alpha_{\pm} = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}. \quad (3.2)$$

The corresponding null-vector then has weight $\Delta = \Delta_{(r,s)} + rs$. $|\chi\rangle$ is known as a null-vector because it has zero norm $\langle \chi | \chi \rangle$, and any correlator involving its associated operator $\chi_{\Delta}(z)$ vanishes. Therefore it can self-consistently be set to zero everywhere. Thinking of $\chi_{\Delta}(z)$ as in some way a special case of a primary operator, we see that any secondaries $[\chi_{\Delta+k}(z)]$ associated with it will also now be zero. In this case the conformal family $[\mathcal{O}_j] \equiv [\mathcal{O}_{(r,s)}]$ will contain ‘fewer’ fields than usual and hence is said to be degenerate. More formally, the representation of the conformal algebra associated to $[\mathcal{O}_{(r,s)}]$ is reducible.

An important result is that any correlator involving an operator from a degenerate family will satisfy a partial differential equation. For instance:

$$\left[\frac{3}{2(2\Delta_{(1,2)} + 1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^{\infty} \left\{ \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right\} \right] \left\langle \mathcal{O}_{(1,2)}(z) \prod_{i=1}^N \mathcal{O}_i(z_i) \right\rangle = 0. \quad (3.3)$$

Degenerate primaries also satisfy the so-called *fusion rule* for their products with other degenerate primaries:

$$\mathcal{O}_{(r_1,s_1)} \mathcal{O}_{(r_2,s_2)} = \sum_{k=|r_1-r_2|+1}^{r_1+r_2+1} \sum_{l=|s_1-s_2|+1}^{s_1+s_2+1} [\mathcal{O}_{(k,l)}], \quad (3.4)$$

where the variable k (l) runs over the even integers if $r_1 + r_2$ ($s_1 + s_2$) is odd, and vice versa. The square bracket here just means that all the operators in that conformal family may be involved in the sum. In the case that $(r_1, s_1) = (1, 2)$ or $(r_1, s_1) = (2, 1)$ then the fusion rules become:

$$\begin{aligned} \mathcal{O}_{(1,2)} \mathcal{O}_{(r,s)} &= [\mathcal{O}_{(r,s-1)}] + [\mathcal{O}_{(r,s+1)}], \\ \mathcal{O}_{(2,1)} \mathcal{O}_{(r,s)} &= [\mathcal{O}_{(r-1,s)}] + [\mathcal{O}_{(r+1,s)}], \end{aligned} \quad (3.5)$$

so $\mathcal{O}_{(1,2)}$ and $\mathcal{O}_{(2,1)}$ are acting as *shift operators*. Notice that the fusion rules (3.5) are closed within the degenerate conformal families¹. This leads to the possibility of a

¹That is, they are truncated from below such that r and s are always positive non-zero and hence only degenerate families contribute.

CFT in which all the primaries are degenerate. If this theory is to be physical then it must, as was discussed earlier, have only real weights. An examination of (3.2) confirms that we are therefore constrained within the domain $c \leq 1$.

In the special case that $\alpha_-/\alpha_+ = -p/q$, where p and q are positive integers (i.e. p/q is rational), something quite remarkable happens to the CFT. Equation (3.2) becomes:

$$\Delta_{(r,s)} = \Delta_0 + A(qr - ps)^2, \quad (3.6)$$

for some number A to be determined. Notice what happens if we write $r = r_0 + np$ and $s = s_0 + nq$ with $r_0 < p$ and $s_0 < q$ in (3.6):

$$\Delta_{(r,s)} = \Delta_0 + A(q(r_0 + np) - p(s_0 + nq))^2 = \Delta_0 + A(qr_0 - ps_0)^2. \quad (3.7)$$

We therefore see that the degenerate primary associated with $r = r_0$ and $s = s_0$ is the same degenerate primary associated with $r = r_0 + np$ and $s = s_0 + nq$ for any integer n . This degenerate primary will have null-operators in its associated conformal family with weights $\Delta = \Delta_{(r_0,s_0)} + (r_0 + np)(s_0 + nq)$. Each value of n will correspond to a different null-operator. So we see that these special theories have not just one null-operator in their degenerate conformal families, but an infinite number of them. These theories are known as *minimal models*.

In a minimal model the correlators will satisfy infinitely many differential equations obtained by the nullification of all of the corresponding null-operators. What is more, the fusion rules (3.5) can be shown to be not only truncated from below, but also truncated from *above*. This means that there are only a finite number of primary operators, $\mathcal{O}_{(r,s)}$ labelled by $(r, s) < (p, q)$, in the theory. However, primaries labelled by (r, s) have the same weights as primaries labelled by $(p - r, q - s)$, so the total number of weights contributing to the closed fusion algebra is $(p - 1)(q - 1)/2$. We encode this symmetry by placing the condition $qr \geq ps$ on (r, s) . Using equation (3.2) we see that the central charge of the minimal theory associated with (p, q) will be given by:

$$c = 1 - \frac{6(p - q)^2}{pq} < 1, \quad (3.8)$$

with α_+ and α_- given by:

$$\alpha_+ = \sqrt{\frac{q}{p}}, \quad \alpha_- = -\sqrt{\frac{p}{q}}, \quad (3.9)$$

which means that the weights of the degenerate primaries (3.2) can be expressed as:

$$\Delta_{(r,s)} = \frac{(rq - sp)^2 - (p - q)^2}{4pq}. \quad (3.10)$$

Let us illustrate the above minimal models by examining the simple example of $(p, q) = (3, 4)$, referred to as the $(3, 4)$ -model. This has central charge equal to a half (i.e the same as for a free fermion) and operators with weights:

$$\Delta_{(1,1)} = \Delta_{(2,3)} = 0, \quad \Delta_{(2,1)} = \Delta_{(1,3)} = \frac{1}{2}, \quad \Delta_{(1,2)} = \Delta_{(2,2)} = \frac{1}{16}. \quad (3.11)$$

This model turns out to describe the critical point of the two-dimensional Ising model, with the three primary fields being identified with the identity ($\Delta = 0$), energy density ($\Delta = 1/2$) and local spin ($\Delta = 1/16$). It turns out that this model is unitary ($c \geq 0$), as are all minimal models in the $(m, m + 1)$ series.

3.2 Minimal String Theory

We can use the principle of general covariance to couple the above minimal models to worldsheet gravity, using the partition function (1.45). To do this we find that we must set $b^2 = p/q$. Recall from Section 1.8 that the dimension of this string theory is effectively² $D = c + 1$, where the ‘1’ comes from the Liouville field, φ . From (3.8) we obtain:

$$D = 2 - \frac{6(p - q)^2}{pq} < 2. \quad (3.12)$$

We have therefore defined string theory in *less than two dimensions*. So the $(2, 3)$ -model corresponds to one spacetime dimension ($D = 1$) (pure gravity³); the $(3, 4)$ -model corresponds to one-and-a-half spacetime dimensions ($D = 3/2$); and the $(1, 2)$ -model naively corresponds to minus-one spacetime dimensions ($D = -1$).⁴ Actually, this last model does not really exist as a conformal field theory because it has no primary operators at all. Later on we will see that it makes sense as a *topological* string theory, having no dynamics. One should not be worried by the occurrence of fractional dimensions here. Rather, one thinks as the dimensionality as the effective

²Though recall that this is strictly only correct for unitary models with $c \geq 0$.

³This is clearly a conformal field theory with no free bosons. Therefore the CFT part of the action is trivial: the only primary operator is $\mathcal{O}_{(1,1)}^0 = \mathcal{O}_{(1,2)}^0$, which can only be the identity.

⁴Though since this model is non-unitary this is not strictly true.

number of degrees of freedom in the target space. So less than one whole dimension is like constraining the string in some way. When we later refer to the string target space we will almost exclusively find ourselves discussing only the Liouville mode itself.

Recall from Section 1.8 that we have to gravitationally dress the conformal operators in order convert them into conformally invariant vertex operators on the worldsheet. Denoting the original undressed primaries as $\mathcal{O}_{r,s}^0$, we obtain the dressed operators, $\mathcal{O}_{r,s}$, via;

$$\mathcal{O}_{r,s} = e^{2\beta_{r,s}\varphi} \mathcal{O}_{r,s}^0, \quad (3.13)$$

where equation (3.10) and conformal invariance (1.52) determines:

$$\beta_{r,s} = \frac{p+q-(rq-sp)}{2\sqrt{pq}}, \quad (3.14)$$

where, as explained in Section 1.8, we have chosen the solution with $\beta_{r,s} \leq Q/2$. Recall from Chapter 1 that the only physical string excitation in $D = 2$ bosonic string theory is the massless closed string ‘tachyon’, $T(\varphi)$.⁵ In minimal bosonic string theory the situation is analogous, although now the ‘tachyon’ has a *positive* mass squared.

3.3 Superminimal Models

There are supersymmetric analogues of the minimal models known as superminimal models. These are again parametrised by two integers (p, q) , although this time p and q must be either both odd and coprime, or both even with $p/2$ and $q/2$ coprime. In the latter case one must also have $(p-q)/2$ odd to ensure modular invariance [15]. Recall from (1.39) that it is more convenient to work with the quantity \hat{c} in supersymmetric theories, rather than the central charge, c , itself. In these superminimal models \hat{c} is given by [16]:

$$\hat{c} = 1 - \frac{2(p-q)^2}{pq}. \quad (3.15)$$

As in the bosonic theories there are conformal primaries, $\mathcal{O}_{r,s}^0$, with $r = 1, \dots, p-1$, $s = 1, \dots, q-1$ and $\mathcal{O}_{p-r, q-s}^0 \equiv \mathcal{O}_{r,s}^0$. This time, however, the operators with $r-s$ even belong to the NS sector, and those with $r-s$ odd belong to the R sector. The

⁵Suppressing the dependence on the time dimension, X^0 .

operators have conformal dimensions given by:

$$\Delta_{(r,s)} = \frac{(rq - sp)^2 - (p - q)^2}{8pq} + \frac{1 - (-1)^{r-s}}{32}. \quad (3.16)$$

We can couple these models to Superliouville theory to give us a set of minimal superstring theories, with dimensions $D = \hat{c} + 1$ in the usual way. Superliouville theory is similar to ordinary Liouville theory, with the analogue of Q in (1.46) given by:

$$Q = \sqrt{\frac{9 - \hat{c}}{2}}. \quad (3.17)$$

The operators of the minimal superstring theories can then be obtained by supergravitationally dressing the operators of the superminimal models. As in the bosonic case, we find that we must set $b^2 = p/q$. An operator that will play a significant role is $\mathcal{O}_{\frac{p}{2}, \frac{q}{2}}^0$. This corresponds to the supersymmetric Ramond ground state and is only present in the (p, q) even models [15]. However, it can only be supergravitationally dressed for one sign of the bulk cosmological constant, $\mu > 0$. This will have important consequences that will become apparent later. Since $\mathcal{O}_{\frac{p}{2}, \frac{q}{2}}^0$ is absent in the (p, q) odd models this means that they break worldsheet supersymmetry.

The spacetime field content of the simplest theory ($\hat{c} = 0$) consists of the NS-NS ‘tachyon’, $T(\varphi)$, and the R-R scalar potential, $C(\varphi)$. This is the $(2, 4)$ model. These theories are purely bosonic in spacetime and so can be identified with the Type 0 theories described in Chapter 1. However, just like in the Type II models, the Type 0 theories also admit various D-branes, providing open string sectors. Although the original closed string theories were purely bosonic, these open string sectors also contain spacetime fermions [2, 17].

3.4 D-Branes in Noncritical and Minimal String Theories

As well as the bulk action, (1.45), we can also include open strings in the usual way by considering boundary Liouville theory coupled to a boundary matter conformal field theory. For pure Liouville theory the boundary action is [18, 19]:

$$S_{\text{boundary}} = \int_{\partial M} ds \left(\frac{Qk}{2\pi} \varphi + \mu_B e^{b\varphi} \right) \hat{g}^{1/4}, \quad (3.18)$$

where k is the boundary curvature also displayed in (1.10), and Q and b are defined in (1.46). The constant μ_B here is known as the *boundary cosmological constant*. This weights boundary lengths in an analogous way to how the bulk cosmological constant weights worldsheet areas.

When translated into the string theory setting, the results of [18, 19] correspond to two possible types of D-brane in the Liouville direction. These arise from the direct product of a Liouville boundary state with the ordinary D-branes from the matter sector, and are known as ZZ and FZZT branes respectively. That the result is a direct product follows from the fact that the matter theory is conformal [15]. Since the string theory is less than two-dimensional, both the ZZ and FZZT branes are stable.

The FZZT branes are semi-infinite, and stretch from the weak coupling regime, $\varphi = -\infty$, to some typical position, $\varphi_0 \propto -\ln |\mu_B|$, set by the boundary cosmological constant. If $\mu_B = 0$ then the FZZT branes extend along the entire Liouville direction to $\varphi = \infty$.

The ZZ branes on the other hand, are localised in the strong coupling regime at $\varphi = \infty$. It turns out that these boundary states can be written as special differences between two FZZT states, which must have the same boundary cosmological constant as the ZZ brane they produce. In minimal bosonic string theory there are only a finite number of distinct ZZ states. For the (p, q) model these are parametrised by two integers (r, s) with $1 \leq r \leq p-1$, $1 \leq s \leq q-1$ and $rq - sp > 0$. Recall from Section 3.1 that the conformal primaries, $\mathcal{O}_{r,s}$, are also labelled with the same restrictions, so one may suspect some connection between the conformal primaries and the ZZ branes. These ideas are explored in [20], where it is shown that the ZZ branes are related to conformal primaries dressed as in (3.13), but now with $\beta_{r,s} > Q/2$ instead of $\beta_{r,s} \leq Q/2$. It turns out that in the bosonic (p, q) model, the (r, s) ZZ brane has the following value of boundary cosmological constant [15]:

$$\mu_B(r, s) \propto \sqrt{|\mu|} (-1)^r \cos \pi s b^2. \quad (3.19)$$

So there are $(p-1)(q-1)/2$ species of ZZ brane. In the Type 0 theories the boundary cosmological constants are given by:

$$\mu_B(r, s) \propto \begin{cases} \sqrt{|\mu|} \cos \frac{\pi}{2}(r + s b^2) & \eta = +1, \\ i \sqrt{|\mu|} \sin \frac{\pi}{2}(r + s b^2) & \eta = -1, \end{cases} \quad (3.20)$$

where $\eta = +1$ (-1) corresponds to the brane being in the NS (R) state for one sign of μ ; and the R (NS) for the other sign of μ (the precise details depend on whether

we are studying the Type 0A theory or Type 0B theory). For $\eta = +1$ (-1) we must have $r - s$ even (odd). For (p, q) even theories there are therefore $[(p-1)(q-1) \pm 1]/2$ ZZ branes in the $\eta = \pm 1$ sector. This implies that one sign of μ has one more NS brane than it has R branes, and vice-versa for the other sign of μ . Recall that the R ground state operator of the (p, q) even models, $\mathcal{O}_{\frac{p}{2}, \frac{q}{2}}^0$, can only be supergravitationally dressed for one sign of μ . It turns out that the extra Ramond ZZ brane is also associated with the $\mathcal{O}_{\frac{p}{2}, \frac{q}{2}}^0$ operator, but exists for the opposite sign of μ , with $\mu_B(\frac{p}{2}, \frac{q}{2}) = 0$. So we expect that for one sign of μ we can have closed string fluxes associated with the R ground state; whereas for the other sign of μ we instead have only D-branes. This will prove important later on. For (p, q) odd on the other hand, there are $(p-1)(q-1)/2$ branes in either sector.

The problem is now to evaluate the full partition function of minimal string theory (1.45) so that we can calculate the n -point functions and solve the theory. Using standard methods this is very non-trivial, and to accomplish it we will have to resort to the use of *matrix models*.

3.5 Discretising The String Worldsheet: Matrix Models

Consider the noncritical string action of (1.44):

$$S[X, g] = \frac{\lambda}{4\pi} \int d^2\sigma \sqrt{g} R + \frac{\mu_0}{8\pi} \int d^2\sigma \sqrt{g} + S_M[X, g], \quad (3.21)$$

and recall that $S_M[X, g]$ denotes the action of the matter section⁶, with μ_0 being the bare bulk cosmological constant. Recalling the discussion at the very end of Section 1.2, we see that we can integrate over the R dependent term in the action to obtain a term proportional to the Euler number the vacuum worldsheet (we will initially only consider closed string worldsheets). Recall that the sum over topologies is implicit in the definition of the partition function. In the case of closed string theory we have:

$$Z = \int \mathcal{D}g \mathcal{D}[\text{matter}] e^{-S[X, g]} = \sum_{h=0}^{\infty} \nu_0^{2h} Z_h, \quad (3.22)$$

⁶We have explicitly included an extra interaction term here for illustrative purposes. Recall from Section 1.8 that the strength of string interactions will also be controlled by the Liouville mode, φ .

where $\nu_0 \equiv e^\lambda$ and Z_h is the partition function for a vacuum worldsheet with a given number of handles, h .

The partition function of (3.22) cannot be evaluated exactly by standard methods. Instead, we proceed by approximation. We break up the worldsheet into n -plaquettes or *simplices*, which are flat regions of a given area of any n -sided shape [21]. For instance, we can approximate the worldsheet by tiling it with triangle-shaped pieces (see Figure 3.1). This process is known as *discretisation* and is similar to the technique used in lattice gauge theories. Fields such as $X^\mu(\sigma_1, \sigma_2)$ will then become discrete values X_i^μ , where i is a label numbering the different plaquettes. In the usual way, derivatives in the action $\partial_\sigma X^\mu(\sigma)$ will become differences $X_i^\mu - X_j^\mu$ between nearest-neighbour plaquettes. If we consider a *triangulation* in terms of equilateral triangles then we can encode the discretised surface in terms of an *adjacency matrix*, an entry C_{ij} of which is unity if plaquettes i and j are neighbours, and zero otherwise. The area of the surface is then proportional to the number of triangles it contains, n . We can use the discrete version of the Gauss-Bonnet theorem to show that the Euler number of such a surface is given by:

$$\chi = v - e - n = 2 - 2h, \quad (3.23)$$

where v is the number of vertices in the triangulation and e is the number of sides. The local curvature at the point i turns out to be proportional to $6 - \sum_j C_{ij}$. So six triangles meeting at a point gives locally flat curvature, whereas five and seven triangles give positive and negative curvature respectively. Since this is an entirely geometrical description of the surface, it is manifestly invariant under diffeomorphisms. Two triangulations are equivalent if their adjacency matrices are identical up to a transposition of points. In the case of pure gravity, $c = 0$, there is no integral over the matter section to perform and the partition function (3.22) becomes:

$$Z = \sum_{h=0}^{\infty} \nu_0^{2h} \sum_{n=0}^{\infty} e^{-\mu_0 n} t_{h,n}, \quad (3.24)$$

where here $t_{h,n}$ is merely the number of ways one can *triangulate* the surface.

Given a triangulation, one can always define its *dual*. This is constructed by drawing a dot at the centre of each triangle and then connecting neighbouring dots (see Figure 3.1). So each dot has three lines joining it. Notice that the result is just like a Feynman diagram with 3-point interactions [22]. For every triangulation there

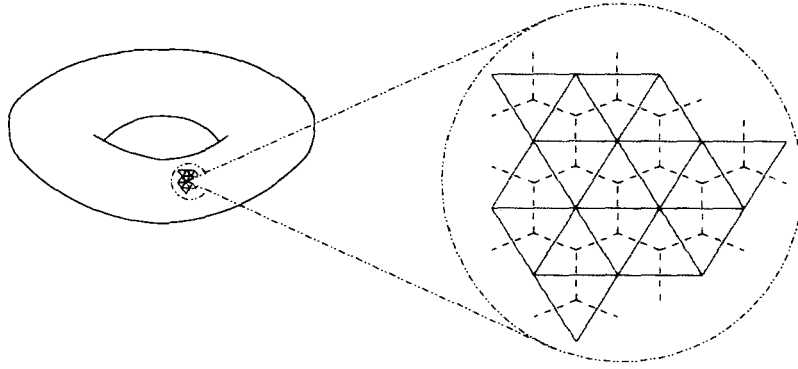


Figure 3.1: A torus worldsheet is triangulated with plaquettes (green triangles). The dual hexagonal lattice (dashed red) forms a Feynman diagram with 3-point interactions.

exists a corresponding Feynman diagram; hence the enumeration of triangulations is equivalent to the enumeration of Feynman diagrams. If we discretise the surface with n -plaquettes then this will correspond to a Feynman diagram with n -point interactions. This problem is tractable via the matrix model developed in [23] and [24].

A matrix model can be thought of as a quantum field theory with a $N \times N$ matrix valued field, M . This field has no arguments however, so the theory is effectively zero-dimensional. For now we will study the model in which M is Hermitian. The partition function for a theory with 4-point interactions (tiling with square plaquettes) is given by:

$$Z_M(N, g) = e^{-F_M} = \int \mathcal{D}M \exp \left[-N \text{Tr} \left(\frac{1}{2} M^2 + g M^4 \right) \right], \quad (3.25)$$

where F_M is a quantity known as the *free energy*. Just like in ordinary quantum field theory, this is just the partition function for *connected* Feynman diagrams only, and so is a quantity of great physical importance. The measure $\mathcal{D}M$ is given by:

$$\mathcal{D}M = \prod_i dM_{ii} \prod_{i>j} dM_{ij} dM_{ij}^*. \quad (3.26)$$

Because the field M_{ij} has two indices associated to it one can think of it as representing different types of particle labelled by (i, j) . Each propagator in the Feynman diagram should therefore be drawn as a double line to represent the propagation of each of its indices. Figure 3.2 shows part of a typical diagram. Because propagators have been replaced by double lines, the resulting graph is often referred to as a *fat Feynman diagram*. The basic propagator and vertex are depicted pictorially in Figure 3.2, and

given mathematically by:

$$\langle M_{ab} M_{cd} \rangle = \frac{\delta_{ad} \delta_{bc}}{N}, \quad \langle M_{ab} M_{cd} M_{ef} M_{gh} \rangle = g N \delta_{bc} \delta_{de} \delta_{fg} \delta_{ha}. \quad (3.27)$$

So it is easy to see that a diagram with P propagators, L loops and V vertices contributes a factor of:

$$g^V \left(\frac{1}{N} \right)^P N^L N^V, \quad (3.28)$$

where for every closed loop one must take a trace over the indices contributing a factor of N (c.f. QCD). This is because closed loops are like virtual particles, which can be any of the N species, corresponding to the inner index of the loop being free. Using the dual map to the original discretisation we see that P , V and L are equal to the dual quantities e , n and v respectively. Using the discrete Euler relation (3.23), we can rewrite (3.25) as:

$$\begin{aligned} Z_M(N, g) &= \sum_{h=0}^{\infty} N^{2-2h} Z_M^{(h)} = \sum_{h=0}^{\infty} N^{2-2h} \sum_{n=0}^{\infty} g^n t_{h,n}, \\ F_M(N, g) &= \sum_{h=0}^{\infty} N^{2-2h} F_M^{(h)} = \sum_{h=0}^{\infty} N^{2-2h} \sum_{n=0}^{\infty} g^n s_{h,n}, \end{aligned} \quad (3.29)$$

where $t_{h,n}$ counts the number of distinct Feynman diagrams with n -vertices that can be drawn on a surface with h handles; $s_{h,n}$ being its connected equivalent. Comparison to (3.24) allows us to make the identification $\nu_0 = 1/N$, $e^{-\mu_0} = g$.

One can ask the question of why we need to define a matrix valued field to generate our triangulations. What is wrong with a standard scalar field theory? The answer is that such diagrams do not have enough structure to fully specify a Riemann surface representing the worldsheet. The additional structure needed is given by widening the propagators to give the ‘fat’ Feynman diagrams. This is because the closed loops representing plaquettes now carry matrix indices that allow unique specification of the locations and orientations of faces [22].

Although at first glance the above formalism is indeed just an approximation, we have in mind from the start the possibility of taking some limit of the theory that will recover the continuum worldsheet. The exact procedure for taking this limit will arise naturally below, but in taking such a limit we must be careful to recover only *universal* physics. By this we mean that the final result should not be an artifact of any particular discretisation that we happen to choose. That is, the final results should

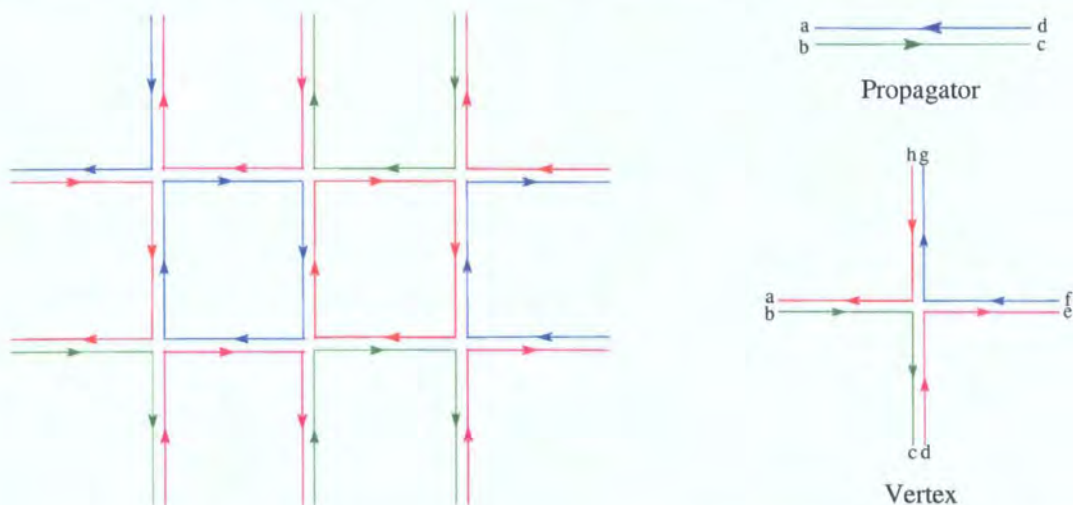


Figure 3.2: Part of a typical discretised worldsheet tiled with square plaquettes; alongside the appropriate propagator and vertex given mathematically in equation (3.27).

be independent of our method of calculation. Any results that are not independent in this way are known as *non-universal* [25].

In the limit that we send the size, N , of the matrix to infinity, there will be many simplifications. We see from (3.29) that in this *large N limit* the partition function will only contain contributions from those diagrams that can be drawn on the surface of a sphere ($h = 0$). For obvious reasons these are known as *planar diagrams*. From the above identification of $\nu_0 = 1/N$, we see that this limit corresponds to sending the string coupling to zero.

To proceed, one diagonalises the matrix M by writing it as $M = U^\dagger \Lambda U$, where U is a unitary matrix diagonalising M , and $\Lambda = \text{diag} \{\lambda_1, \lambda_2, \dots, \lambda_N\}$ is the diagonalised matrix. This converts the partition function to an integral over eigenvalues λ_i , allowing us to write:

$$Z_M(N, g) = \int_{-\infty}^{\infty} \prod_i d\lambda_i \Delta^2(\lambda) dU \exp \left[-N \sum_i \left(\frac{1}{2} \lambda_i^2 + g \lambda_i^4 \right) \right] \quad (3.30)$$

Here, $\Delta^2(\lambda)$ is the Jacobian for the change of variables known as the *Vandermonde* determinant. It is given by:

$$\Delta^2(\lambda) = \prod_{i < j} (\lambda_i - \lambda_j)^2. \quad (3.31)$$

Proof of this result, as well as similar Jacobians for different types of matrix (e.g. unitary, orthogonal etc) can be found in [26]. It is easy to show that $\Delta(\lambda)$ can be

written as the determinant of a $N \times N$ matrix, $T(N)$, whose i, j entry is λ_i raised to the power j . The dU integral in (3.30) merely contributes an overall g -independent factor to the partition function; it can therefore be ignored. Notice that (3.30) resembles an ensemble of charges in one dimension with positions λ_i . These charges move in an external quartic potential:

$$V_{ext}(\lambda) = \frac{1}{2}\lambda^2 + g\lambda^4. \quad (3.32)$$

By exponentiating the Vandermonde determinant we see that these particles interact via a logarithmic repulsion:

$$V_{rep}(\lambda_i, \lambda_j) = \ln |\lambda_i - \lambda_j|. \quad (3.33)$$

This system is known as a *Dyson gas*. It is clear from the symmetry of the problem that the eigenvalues will organise themselves evenly around zero. In the planar limit, $N \rightarrow \infty$, the position coordinate, λ_i , may be replaced by a continuous field $\lambda(x) \equiv \lambda(i/N)$, with density $\rho(\lambda) = dx/d\lambda$. In Section C.1 we will calculate the free energy on the sphere explicitly. Here we will just state the results. It turns out that the eigenvalues distribute themselves according to:

$$\rho(\lambda) = \frac{1}{\pi} \left(\frac{1}{2} + 4ga^2 + 2g\lambda^2 \right) \sqrt{4a^2 - \lambda^2}, \quad |\lambda| \leq 2a, \quad (3.34)$$

where $a(g)$ can be found via:

$$12ga^4 + a^2 - 1 = 0 \quad (3.35)$$

The free energy on the sphere is given perturbatively (for small g) by:

$$F_M^{(0)}(g) - F_M^{(0)}(0) = 2g - 18g^2 + 288g^3 - 6048g^4 + O(g^5). \quad (3.36)$$

The interpretation of this series is that each power of g present will correspond to an extra loop in the vacuum diagram, with the g^1 term being the two-loop diagram. The numerical factors then count the number of different possible diagrams including their symmetry factors at each order. The results of this matrix model calculation agree exactly with the answers generated by more primitive methods.

The above techniques are very elegant, but unfortunately only give us information about the string partition function on the sphere. To gain information about higher orders in string perturbation theory we apparently need to study the matrix model

at finite values of N . In practice this is very difficult. However, we will see shortly that we can both take the computationally-friendly large- N limit *and* evaluate the full string partition function. The technology that will allow us to do this is known as the *double scaling limit*.

3.6 The Double Scaling Limit

Notice from (3.35) that $a^2(g)$ has a double root at $g_c = -1/48$. This defines the radius of convergence for the perturbation series (3.36). Figure 3.3 shows the configuration of the Dyson gas at both $g = 0$ and $g = g_c$. We see that at $g = 0$ the eigenvalues sit in the potential as a *Wigner* semi-circular distribution. For $g > 0$ this semicircle flattens out, becoming oblate; for $g < 0$ the semicircle becomes prolate. In all cases the tails of the distribution fall off like $\sqrt{4a^2 - \lambda^2}$.⁷ However, when $g = g_c$ we see that the tails of the distribution gain an extra zero and fall off like $(4a^2 - \lambda^2)^{3/2}$.⁸ Physically this corresponds to the relative repulsion of the eigenvalues being on the verge of overcoming the external potential and ‘leaking’ out. For $g < g_c$ we see that the eigenvalue density no longer makes any physical sense: the eigenvalues have escaped from the potential. Near $g = g_c$ the vacuum energy has a singularity of the form [22]:

$$F_M^{(0)}(g) - F_M^{(0)}(0) \sim (g_c - g)^{5/2} (1 + O(g - g_c)) + \text{analytic terms}, \quad g \sim g_c, \quad (3.37)$$

up to a multiplicative constant. Here we have singled out analytic terms because they will correspond to non-universal contributions after we have taken the double scaling limit [21]. By making use of (3.29) we see that the expectation value for the area of the discretised worldsheet made up of n plaquettes, $\langle n \rangle$, is clearly given by:

$$\langle n \rangle = \langle A \rangle = \frac{1}{F_M^{(0)}} \sum_{n=0}^{\infty} n g^n s_{h,n} = \frac{\partial}{\partial g} \ln F_M^{(0)}(g) \sim \frac{1}{g - g_c}, \quad (3.38)$$

so we see that in the limit that $g \rightarrow g_c$, the average number of plaquettes constituting the discretised worldsheet diverges. If we combine this with letting the size of each individual plaquette tend to zero, then we see that we should recover the continuum worldsheet.

⁷So the right-hand tail will fall off like $\sqrt{2a - \lambda}$.

⁸So the right-hand tail will now fall off like $(2a - \lambda)^{3/2}$.

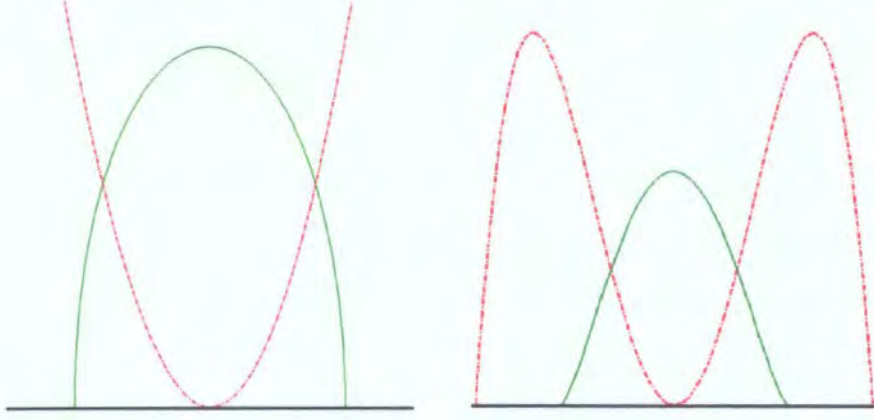


Figure 3.3: The eigenvalue distribution at $g = 0$ is plotted (left, green), along with the shape of the corresponding external potential (left, dotted red). The same curves are plotted in the right-hand figure, but for $g = g_c$. Note that the graphs are not drawn to scale.

Recall that in moving from the string theory to the matrix model we associated $g = e^{-\mu_0}$. Correspondingly we set $g_c = e^{-\mu_c}$. So we see that recovering the continuum limit in a way that recovers the universal physics is a matter of tuning the bulk cosmological constant to a particular critical value, $\mu_0 = \mu_c$. We therefore see from (3.39) that we should expect the following behaviour at criticality:

$$F_M^{(0)}(\mu_0) - F_M^{(0)}(0) \sim (\mu_c - \mu_0)^{5/2} (1 + O(\mu_c - \mu_0)) + \text{analytic terms}, \quad \mu_0 \sim \mu_c. \quad (3.39)$$

Here we have used the result that:

$$e^{\mu_0}(g - g_c) = (1 - e^{\mu_0 - \mu_c}) = \mu_c - \mu_0 + O(\mu_c - \mu_0)^2. \quad (3.40)$$

In fact, using the results of Section 1.8 it can be shown that for any noncritical string theory we expect the following critical behaviour:

$$F_M^{(0)}(\mu_0) - F_M^{(0)}(0) \sim (\mu_c - \mu_0)^{2-\gamma_{str}} (1 + O(\mu_c - \mu_0)) + \text{analytic terms}, \quad \mu_0 \sim \mu_c, \quad (3.41)$$

where γ_{str} is defined in (1.51). We will use this result shortly.

As was alluded to above, the physics should not depend on the way we discretise the worldsheet. We can equally well tile the worldsheet with various combinations of

plaquettes corresponding to the general matrix model potential:

$$V(\lambda) = \sum_{i=1}^p g_i \lambda^i, \quad (3.42)$$

with couplings g_i . It can be shown that this yields the same critical behaviour on the sphere (3.39) under tuning of g_p to some critical value. This universality extends to the partition function on any higher genus surface. In the Dyson gas picture this corresponds to critical behaviour that only cares about the eigenvalues living in the tail of the distribution. The other details of the potential are irrelevant.

However, in the same way that we tuned $g \equiv g_4$ in the above, we now have $p + 1$ free parameters in the theory to tune. We can adjust the g_i to accumulate any number of zeroes at either tail of the eigenvalue distribution. Each will give new critical behaviour, and will correspond to a new continuum string theory. The critical behaviour of the model turns out to be controlled by the end of the eigenvalue distribution with the higher number of extra zeroes, $m - 1$, say. The critical behaviour of the free energy of the m -th model can be shown to behave as $(\mu_c - \mu_0)^{2+1/m}$ [22]. That is, $\gamma_{str} = -1/m$. Using (1.51) we see that the expected central charge of this string theory is:

$$c = \frac{m^2 + m - 6}{m(m + 1)} = 1 - \frac{6}{m(m + 1)}. \quad (3.43)$$

Further comparison with (3.8) strongly suggests that this is the $(m, m + 1)$ minimal string theory, which is the unitary series discussed above. We will see later that this cannot be the case. The theories obtained by such tuning of the g_i are known as *Kazakov's multicritical points*. The picture one should have is of a space spanned by the g_i . Arbitrary positions in this space will invariably lead to pure gravity. However, certain submanifolds in this space will coincide with tuning the theory to the multicritical points. The lower the dimension of the submanifold, the higher the multicritical point.

Comparison of (3.29) and (3.39) shows that although we have isolated the sphere contribution $F_M^{(0)} \sim (\mu_c - \mu_0)^{5/2}$, we have $F_M \rightarrow N^2(\mu_c - \mu_0)^{5/2}$, which diverges in the large N limit. We saw above that the correct continuum physics arises in the limit $N \rightarrow \infty$ and $g \rightarrow g_c$. When we take this limit we need to make sure that we recover a finite value for the partition function F_M . To do this we choose the following scaling:

$$\mu_c - \mu_0 = \mu \delta^4, \quad \nu_0 \equiv \frac{1}{N} = \nu \delta^5, \quad (3.44)$$

where μ and ν are the *renormalised* quantities corresponding to $\mu_c - \mu_0$ and ν_0 respectively; δ is a parameter that we will send to zero to recover the continuum. It can be thought of as in some way representing the edge length of the plaquettes. Taking $g \rightarrow g_c$ and $N \rightarrow \infty$ in this correlated way is known as the *double scaling limit* [21]. As we send $\delta \rightarrow 0$ the sphere contribution becomes:

$$F_M \rightarrow \frac{(\mu\delta^4)^{5/2}}{\nu^2\delta^{10}} = \frac{\mu^{5/2}}{\nu^2} \equiv \frac{1}{g_s^2}, \quad (3.45)$$

which is finite in terms of some new dimensionless parameter $g_s \equiv \nu/\mu^{5/4}$. So we have recovered the universal sphere physics in terms of the suggestively named parameter g_s . However, in the double scaling limit the sphere term no longer dominates the free energy, F_M , and so we will now be able to recover the full all-genus partition function.

So in this limit we are justified in saying that the matrix model is dual to the string theory. One interpretation of this is that it is an example of a holographic principle, similar to the one underpinning the AdS/CFT correspondence. In that case, a field theory in four dimensions is dual to a string theory in five dimensions. In the matrix model case, a matrix field theory in zero dimensions is related to a string theory living in effectively one dimension, the Liouville mode. This is an interesting parallel that relates to an interpretation of the matrix model as a worldvolume theory of unstable D0 branes [27].

To proceed we will need the elegant methods of orthogonal polynomials. These, and the subsequent calculation, are explained in Section C.2. The final result is the following *string equation* in terms of a function $u(z)$:

$$-\frac{\nu^2}{3} \frac{\partial^2 u}{\partial z^2} + u^2 - z = 0, \quad (3.46)$$

where z is a new variable that is naively equivalent to μ . Equation (3.46) is known as the Painlevé I equation of the mathematical literature. The free energy can be obtained from the following relationship:

$$u(z) = -\nu^2 \frac{\partial^2 F_M}{\partial z^2}. \quad (3.47)$$

The string equation (3.46) will in principle give us the full partition function of pure gravity at all orders in perturbation theory and for all values of z . However, in practice it is plagued by non-perturbative instabilities: $u(z)$ has poles that are physically worrisome [28]. Let us ignore this for now and search for perturbative solutions at

weak coupling. Recall that the string coupling is related to the parameter ν , which from the discussion below (3.45) we expect should be eventually combined with z to form the parameter $g_s = \nu/z^{5/4}$. We therefore expect large z to correspond to weak coupling. Expanding perturbatively we find:

$$u(z) = \sqrt{z} - \frac{\nu^2}{24z^2} - \frac{49\nu^4}{1152z^{9/2}} + \dots, \quad (3.48)$$

which we can integrate twice to obtain the free energy:

$$F_M = -\frac{4z^{5/2}}{15\nu^2} - \frac{1}{24} \ln|z| + \frac{7\nu^4}{1140z^{5/2}} + O(z^{-7}). \quad (3.49)$$

This can be rewritten as:

$$F_M = -\frac{4}{15}g_s^{-2} - \frac{1}{24} \ln|z| + \frac{7}{1140}g_s^2 + O(g_s^4), \quad (3.50)$$

so this is just the expansion of the free energy in terms of vacuum diagrams weighted by g_s^{2h-2} , where h is the number of handles. So the first term in the above expansion is the sphere, the second the torus, the third the double torus etc. So g_s is playing the role of the *renormalised string coupling* here. One should not be worried by the appearance of the $\ln|z|$ term in the torus contribution. It represents the volume of the Liouville direction. Morally this is still a g_s^0 term because it has no powers of ν . The interpretation of the logarithm comes from the realisation that z is just the renormalised cosmological constant of the string theory. It can be shown that all the higher terms in the free energy g_s^h , $h > 1$, have positive definite coefficients. Differentiating the series for $u(z)$ once, we can give all the coefficients in the perturbative expansion the same sign. This is a sign that the model is unitary [3]. This is as expected for pure gravity, because the $(2, 3)$ model is a member of the $(m, m+1)$ unitary series discussed in Section 3.1.

As well as perturbatively, we can try to get some measure of the non-perturbative physics of the string equation by considering a small perturbation, $j(z)$, to a solution, $u(z)$, of (3.46). We have:

$$-\frac{\nu^2}{3} \frac{\partial^2 u}{\partial z^2} - \frac{\nu^2}{3} \frac{\partial^2 j}{\partial z^2} + u^2 + 2uj + j^2 - z = 0. \quad (3.51)$$

We now use the fact that $u(z)$ is itself a solution of the string equation; and assume $j(z)$ is extremely small. Writing $j(z) = e^{p(z)}$ we find:

$$-\frac{\nu^2}{3} \left[\left(\frac{\partial p}{\partial z} \right)^2 + \frac{\partial^2 p}{\partial z^2} \right] + 2u = 0. \quad (3.52)$$

Since $u(z) \sim \sqrt{z}$ we can solve this equation at leading order and we find $p(z) \sim \pm 4\sqrt{6}z^{5/4}/5\nu$. Since $j(z)$ is small we should take the negative sign for the square root. This yields:

$$j(z) \sim e^{-\frac{4\sqrt{6}}{5g_s}}. \quad (3.53)$$

which is the typical strength of non-perturbative effects in string theory. This is known as the WKB approximation and was first derived in [29]. In quantum field theory we would typically expect non-perturbative instanton effects to scale like e^{-1/g^2} , where g is the coupling of the theory. In string theory we typically find that the non-perturbative effects scale like e^{-1/g_s} . This turns out to be a characteristic feature of the D-branes in string theory, indicating their non-perturbative nature.

3.7 Minimal String Theory and The KdV Hierarchy

Let us rewrite the string equation of pure gravity ($m = 2$), (3.46), in the following form⁹:

$$\mathcal{R}[u] = \tilde{t}_2 R_2[u] - z = 0, \quad (3.54)$$

$$\text{with} \quad R_2[u] = 3u^2 - u'', \quad \tilde{t}_2 = \frac{1}{3}. \quad (3.55)$$

It can be shown that the string equations of the higher m multicritical points can be written in the form:

$$\mathcal{R}[u] = \sum_{k=1}^m \tilde{t}_k R_k[u] - z = 0, \quad (3.56)$$

where \tilde{t}_{m-1} can be set to zero by making various rescalings (it corresponds to a boundary operator [4]), but the lower \tilde{t}_k are dimensioned parameters, the significance of which will become clear later. They arise as the double scaled remnants of the free parameters in the matrix model potential that we tuned to achieve multicriticality.

⁹In what follows we will often use the following shorthand notation: a prime will denote either $\partial \equiv \partial/\partial x$ or $d \equiv \nu\partial/\partial z$ depending on the context (which should always be self-evident). This will allow us to suppress ν in many of our equations.

The $R_k[u]$ are differential polynomials in $u(z)$; the first few are:

$$\begin{aligned} R_0 &= \frac{1}{2}, \quad R_1 = -u, \quad R_2 = 3u^2 - u'' \\ R_3 &= 20u'u'' + 10uu''' - u^{(5)} - 30u^2u'. \end{aligned} \quad (3.57)$$

They also obey the following recursion relation:

$$R'_{k+1} = (d^3 - 4ud - 2u') R_k. \quad (3.58)$$

The above formalism may now be seeming a little familiar. This is because we have recovered the KdV hierarchy of subsection (2.3.1). A perusal of that subsection confirms that the KdV hierarchy can be written as (2.25):

$$\alpha_k \frac{\partial u}{\partial t_k} = R'_{k+1}. \quad (3.59)$$

We therefore see that the physics of an entire family of string theories is encoded by the KdV integrable hierarchy.

We earlier conjectured that the m -th multicritical model would correspond to the unitary $(m, m+1)$ minimal model coupled to Liouville theory. This is supported by the observation that the leading order behaviour of $u(z)$ is proportional to $z^{1/m}$ at large z . However, when one computes the large z behaviour of the string equation $R_3[u] - z = 0$, one finds a problem. The terms in the perturbative expansion are no longer of the same sign, that is, the model is not unitary at all. It is not the $(3, 4)$ Ising model as expected. The model in fact describes the so-called *Yang-Lee edge singularity* [28, 30], which is actually the $(2, 5)$ model. This is somewhat worrying because the expected critical exponent of the $(c = -22/5)$ $(2, 5)$ model using the formula (1.51):

$$\begin{aligned} \gamma_{str} &= 2 - \frac{Q}{b(0)} = \frac{1}{12} \left(c - 1 - \sqrt{(1-c)(25-c)} \right), \\ Q &\equiv \sqrt{\frac{25-c}{6}}, \quad b(0) = \frac{1}{\sqrt{24}} (\sqrt{25-c} - \sqrt{1-c}), \end{aligned} \quad (3.60)$$

is $\gamma_{str} = -3/2 \neq -1/3$. Notice that we have relabelled b from (1.51) as $b(0)$. This is because we have been assuming the operator of lowest dimension in the theory is controlled by the cosmological constant, μ . That is, we have been assuming that z and μ control the same operator, which is true in pure gravity. However, it is speculated in [28] that the coupling of the model to Liouville theory could mean that the lowest

dimension operator in the theory has a negative dimension. This drastically changes the global properties of the theory. The parameter z in the string equation will then be associated with this operator, and no longer will it be equivalent to μ . In this situation the formulae given in (3.60) should be altered by replacing $b(0)$ with $b(\Delta_N)$, where Δ_N is the dimension of the most negative operator. One finds:

$$b(\Delta_N) = \frac{1}{\sqrt{12}}(\sqrt{25 - c} - \sqrt{1 - c + 24\Delta_N}). \quad (3.61)$$

In this respect we can arrange matters such that $\gamma_{str} = -2/(p + q - 1)$ for the (p, q) model. If we do this then the $(2, 5)$ model will have the correct critical exponent, $\gamma_{str} = -1/3$.

To recap, the $m = 2$ critical point describes the $(2, 3)$ model and the $m = 3$ multicritical point describes the $(2, 5)$ model. The natural conjecture is therefore that the m -th multicritical point corresponds to the $(2, 2m - 1)$ minimal model coupled to Liouville gravity. As we have discussed before, for these non-unitary models the central charge takes negative values. This would naively mean that they correspond to string theories in less than one dimension. However, the results of [4, 5, 31] give good arguments to suggest that these particular models are effectively one-dimensional, which means that spacetime is effectively just the Liouville mode.

It remains to explain the role of the lower \tilde{t}_k ($1 \leq k \leq m - 1$) in the string equation (3.56). The natural interpretation of these is that they control other operators in the theory. Recall from Section 3.1 that the (p, q) model has $(p - 1)(q - 1)/2$ primary operators $\mathcal{O}_{(r,s)}$ for $(r, s) < (p, q)$, with $\mathcal{O}_{(r,s)} = \mathcal{O}_{(p-r, q-s)}$. So in the case of the $(2, 2m - 1)$ series we have $m - 1$ conformal primaries. Luckily we also have $m - 2$ nontrivial parameters \tilde{t}_k , plus the extra parameter z . It can be shown that these parameters do indeed control the primary operators of the theory [4]; remarkably, it transpires that we can compute all the correlation functions of the string theory using the following simple formula [32]:

$$\left\langle \prod_i \mathcal{O}_{k_i} \right\rangle = \prod_i \frac{\partial}{\partial \tilde{t}_{k_i}} F_M, \quad (3.62)$$

where we have labelled the conformal primaries by their associated \tilde{t}_k , and have labelled z as \tilde{t}_0 .

It was mentioned above that the string equation (3.46) has non-perturbative problems, specifically the appearance of an infinite number of poles on the negative z axis.

We can see the origin of this problem from the perturbative expansion (3.48). This is only real-valued at large positive values of z , and is complex at large negative z . So it is not possible to define a perturbative expansion at large negative z . This is a sign of underlying instability. In general, the large- z expansion of the m -th string equation leads with $u(z) \sim z^{1/m}$. So the appearance of poles would be expected in at least all the even m models. Numerical studies have shown this to be the case, so we see that string equations of the Hermitian matrix models have severe non-perturbative difficulties. In an attempt to overcome these problems, other matrix models can be studied. One important example is the complex matrix model.

3.8 The Complex Matrix Model and Type 0A Minimal Superstring Theories

The complex matrix model is defined by the following partition function [33, 34]:

$$Z_N = \int dM e^{-\frac{N}{\gamma} \text{Tr}[V(MM^\dagger)]}, \quad (3.63)$$

where M is an $N \times N$ complex matrix and γ is a constant that proves useful when tuning the model to criticality (see Section C.2). We can proceed in the usual manner, by diagonalising the matrix M in terms of complex eigenvalues λ_i . However, the matrix M only appears in the combination MM^\dagger in the partition function. So instead of an ensemble of complex-valued eigenvalues, λ_i , we can instead deal with the eigenvalues, y_i , of MM^\dagger , which are real-valued and greater than or equal to zero. We find that we can write the partition function in the following form:

$$Z_N \sim \int_0^\infty \prod_{i=1}^N dy_i \Delta^2(y) e^{-\frac{N}{\gamma} V(y_i)}. \quad (3.64)$$

This is identical in form to the partition function of the Hermitian matrix model (3.30), except now the integral only covers the half-line. Accordingly, we can once again think of these eigenvalues as a Dyson gas in an external potential $V(y)$. This time however, there is a wall at $y = 0$ that the eigenvalues are prevented from penetrating. Recall that it is the tails of the eigenvalue distribution that are important in the context of the double scaling limit. If an end of the eigenvalue distribution is away from the wall then the double scaling limit associated to it will be identical to that found via the

Hermitian matrix model. So the only new critical behaviour will be found when the eigenvalue distribution pushes up against the wall.

The double scaling analysis can be carried out in much the same manner as in the Hermitian matrix model. One finds the following string equation, again in terms of the function $\mathcal{R}[u]$ defined in (3.56):

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = 0. \quad (3.65)$$

Notice that $\mathcal{R}[u] = 0$ is still a solution of this equation. So the perturbative string expansions of the $(2, 2m - 1)$ models can still hold at large $-z$. However, there is now also a second perturbative expansion, which we will write below. This expansion turns out to be valid at large negative z , and so it seems as if we may be able to find solutions without poles. We will see below that this is indeed borne out numerically.

We can generalise the complex matrix model by making the matrix M rectangular. That is, we make it a matrix of size $N \times (N + \Gamma)$, where Γ is some constant. This gives:

$$Z \sim \int_0^\infty \prod_{i=1}^N dy_i y_i^\Gamma \Delta^2(y) e^{-\frac{N}{\gamma} V(y_i)}. \quad (3.66)$$

This rectangular matrix model was first studied in [35, 36], and yields the following string equation:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2 \Gamma^2. \quad (3.67)$$

Although from the complex matrix model it appears that the constant Γ is restricted to be an integer, it can also be obtained in various other ways [37, 38], none of which poses this $\Gamma \in \mathbb{Z}$ restriction.

In the case of the simplest model, $\mathcal{R}[u] = u(z) - z$, this string equation yields the following large z expansions¹⁰:

$$u = z + \frac{\nu\Gamma}{z^{1/2}} - \frac{\nu^2\Gamma^2}{2z^2} + \frac{5}{32} \frac{\nu^3\Gamma(4\Gamma^2 + 1)}{z^{7/2}} + \cdots \quad \text{as } z \rightarrow +\infty, \quad (3.68)$$

$$u = \frac{\nu^2(4\Gamma^2 - 1)}{4z^2} + \frac{\nu^4(4\Gamma^2 - 1)(4\Gamma^2 - 9)}{8z^5} + \cdots \quad \text{as } z \rightarrow -\infty, \quad (3.69)$$

¹⁰When we take the square root of Γ^2 we will choose the positive sign. The consequences of this will be discussed in Chapter 5.

which correspond to the following free energies:

$$F = \frac{1}{6}g_s^{-2} + \frac{4}{3}\Gamma g_s^{-1} + \frac{1}{2}\Gamma^2 g_s^0 \ln z + \frac{1}{24}\Gamma(4\Gamma^2 + 1)g_s^1 + \cdots \quad \text{as } z \rightarrow +\infty, \quad (3.70)$$

$$F = -\left(\Gamma^2 - \frac{1}{4}\right)g_s^0 \ln |z| + \frac{1}{96}(4\Gamma^2 - 1)(4\Gamma^2 - 9)g_s^2 + \cdots \quad \text{as } z \rightarrow -\infty, \quad (3.71)$$

where we have again defined a dimensionless string coupling $g_s = \nu/z^{3/2}$. An interpretation can again be made in terms of basic string worldsheets for these expansions¹¹. However, now we must modify the Euler number to $\chi = 2 - 2h - b$, where b is some number of boundaries. So, a worldsheet with b boundaries appears to come with a factor of Γ^b , where Γ was originally believed to be some open string coupling parameter. However, with the discovery of D-branes, one can interpret Γ in a new way, as the number of some species of background D-branes in the theory. In [31] these equations were identified as giving the free energies of the $(2, 4m)$ superconformal models coupled to Superliouville theory. These are the Type 0A superstring theories of Section 1.6. The simplest model, with $\mathcal{R}[u] = u(z) - z$, corresponds to the $(2, 4)$ model: pure supergravity ($\hat{c} = 0$). In the same paper, the D-branes associated with Γ were identified as background ZZ branes.

At large negative z it is clear from (3.71) that the free energy contains only surfaces with an even Euler number. The interpretation of this regime is that, instead of corresponding to a number of D-branes, Γ instead represents the insertions of half-units of R-R flux. These fluxes are associated with the R-R scalar, $C(\varphi)$, and are entirely analogous to magnetic fluxes in ordinary electromagnetism. They correspond to valid backgrounds in which closed strings can propagate. In the worldsheet picture they correspond to the insertion of R-R vertex operators into the worldsheet. For example, the first term in (3.71) is a sphere with two R-R punctures. This leads to the idea of a geometric transition [39, 40] whereby in one weak coupling regime we have open strings, closed strings and D-branes; whereas in the other we have only closed strings and fluxes. Recall from Section 3.4 that the R ground state operator of the underlying conformal field theory should only exist for one sign of z (which

¹¹One slight subtlety is that the sphere term in (3.68) is analytic in z . If one performs the matrix model analysis carefully then one is led to suspect that analytic terms are non-universal in nature. Therefore this sphere term should be treated with suspicion.

for the $(2, 4)$ model is equivalent to μ), whereas the ZZ branes should only exist for the opposite sign. It is clear that these observations are realised by the results of this section. It turns out that this flux-brane transition is common to all the members of the $(2, 4m)$ series, because they each permit a negative z expansion with $u(z) \sim \nu^2(4\Gamma^2 - 1)/4z^2$ leading order behaviour. So remarkably, the relatively simple string equation (3.67) can in principle calculate the partition function to all orders in perturbation theory and yet still demonstrate some highly non-trivial phenomena.

We can do better than that though, because (as promised) it is possible to solve the string equation non-perturbatively for all values of z (all values of the string coupling) in a way that smoothly interpolates between the boundary conditions at large positive z and those at large negative z . This is shown in Figure 3.4 for various values of Γ . The dependence of the solutions on Γ will be studied in much more detail later on.

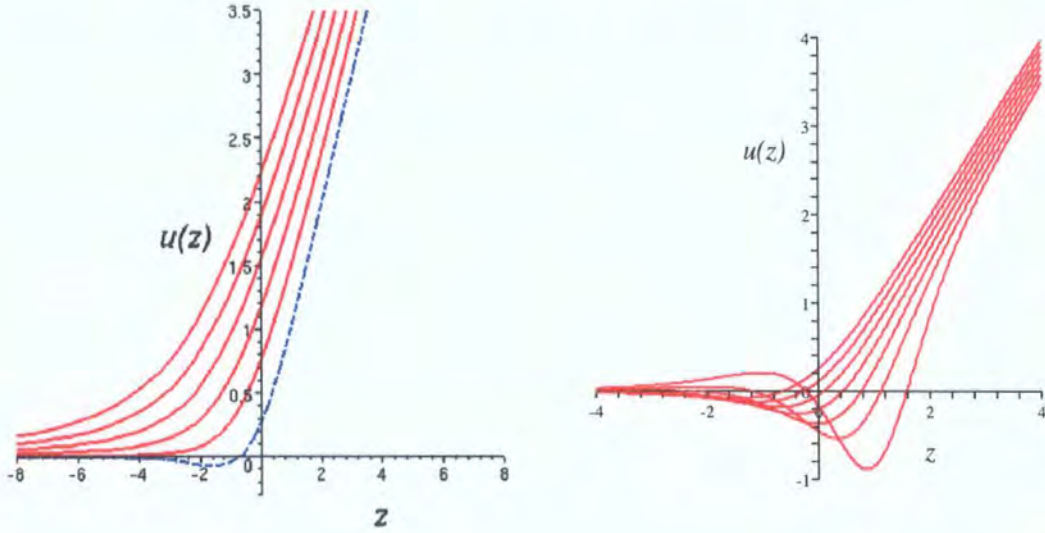


Figure 3.4: Numerical solutions to equation (3.67) for $u(z)$ in the case of pure supergravity ($\Gamma = 0$ is shown in dashed blue): (*l*) cases of positive integer Γ ; (*r*) some cases of $-1 < \Gamma < 0$.

At this point one may ask why there is no analogue of the parameter Γ in the bosonic $(2, 2m - 1)$ series associated with the $\mathcal{R}[u] = 0$ string equation. These theories also permit ZZ branes as we discussed in Section 3.4. The reason for the absence of a parameter that counts such branes is that in the Type 0 models the background ZZ branes are like charged instantons, whereas in the bosonic theory they are uncharged. This means that their contributions must be summed over in the latter case [31], and

so there we are not at liberty to add an arbitrary number of them.

Recall from Section 2.4 that the KdV hierarchy is strongly associated with the following Schrödinger equation:

$$Q\psi \equiv (\partial^2 - u)\psi = \tilde{\lambda}\psi, \quad (3.72)$$

It was from this equation that the Miura map was derived, which related solutions of the KdV hierarchy to those of the mKdV hierarchy. One might wonder whether this Schrödinger equation plays a role in the string theory context. This is indeed the case, and in [41] it was suggested that the Baker function, ψ , corresponds to the partition function of a probe FZZT brane in the $(2, 2m - 1)$ bosonic theories, with string equations $\mathcal{R} = 0$, discussed above. In the context of the underlying Hermitian matrix model the constant $\tilde{\lambda}$ arises as the double-scaled remnant of the eigenvalue distribution. However, in terms of FZZT physics it was conjectured to be the boundary cosmological constant, μ_B , on the probe brane. Evidence for the suggestion came from comparison to known results for the FZZT disc amplitude. We will later see how the worldsheet physics of this result can be expressed in a far more intuitive form, via the Miura map. We will also explore the role of the Schrödinger equation in the $(2, 4m)$ Type 0 theories.

In this chapter we have seen how the physics of the Type 0A $(2, 4m)$ minimal string theories is underpinned by both matrix models and integrable hierarchies. In the next chapter we will attempt to generalise these models to incorporate unoriented string worldsheets. Before doing this, let us ask the natural question of whether the Type 0B $(2, 4m)$ theories can also be related to an integrable system and a matrix model. This is indeed the case: it turns out that the theories are underpinned by both the unitary matrix model and the Zakharov-Shabat integrable hierarchy. This connection will be the subject of Section C.3.

Chapter 4

Unoriented Minimal Type 0 Strings

In this short chapter we will define a family of theories whose perturbative expansions can be naturally interpreted as unoriented Type 0A strings, in the $(2, 4m)$ superconformal minimal model backgrounds studied in Chapter 3. Just like in the oriented case studied there, the models have a well-defined non-perturbative regime. We will compare our results to those already in the literature [42], arising from a study of self-dual unitary matrix models in the double scaling limit. We argue that it is natural to identify this system as the pure supergravity case of unoriented Type 0B string theory.

Let us start by briefly explaining what unoriented string theories are, and how they are constructed [43]. In any string theory we can define a worldsheet parity operator, Ω , that acts as $\Omega : X^\mu(\sigma, \tau) \mapsto X^\mu(\ell - \sigma, \tau) = X^\mu(\sigma, \tau)$. For closed strings this just interchanges left-moving and right-moving coordinates. Operating with Ω twice clearly gives the identity ($\Omega^2 = 1$), and so the eigenvalues of Ω are correspondingly ± 1 . If a theory has the same chirality for its left- and right-movers then its spectrum is invariant under Ω . Such theories include the closed bosonic string, Type 0A, Type 0B and Type IIB [44]. Type IIA is not invariant under Ω . If Ω is a symmetry then it can be accordingly *gauged*, by imposing $\Omega = +1$ on the states. This means that states related by left-right exchange are considered equivalent and, correspondingly, the resultant theory is a quotient of the original theory by Ω . It is obvious that not all the states will survive this projection and, as such, the quotient theory will have a reduced spectrum.

However, there is another important consequence of gauging Ω . In the unquotiented theory all the worldsheets contributing to the partition sum were *oriented*.

That is, if we assign an arrow to point either clockwise or anti-clockwise around the string, then this initial orientation will be maintained as the string evolves. Ω acts to reverse the sign of this arrow. Accordingly, since the gauging process instructs us to make strings with opposite orientations equivalent, we must therefore contend with the possibility that the string can change its orientation as it evolves. This will lead to *unoriented* surfaces such as the projective plane and the Klein bottle contributing to the partition sum. This theory is therefore known as an *unoriented* string theory. A similar process is possible with open string theories, and this will lead to surfaces such as the Möbius band appearing in the partition function. We will see this explicitly below. The process of gauging Ω can be extended in many ways such as by combining with various orbifold projections¹. These quotients are often given the generic name of *orientifolds*, and they lead to many important phenomena such as *orientifold planes* (see below).

4.1 Unoriented Bosonic Minimal String Theory

In order to define an unoriented sector for the Type 0A theories it will be helpful to review how this sector is defined for the bosonic $(2, 2m - 1)$ theories, with string equations $\mathcal{R} = 0$. Recall from (3.46) that the string equation of the $(2, 3)$ ($c = 0$) theory is Painlevé I:

$$-\frac{1}{3}u'' + u^2 = z, \quad (4.1)$$

We recall that this equation resulted from the double scaled Hermitian matrix model, with free energy given by $F'' = -u$.

In [45–48], symmetric matrix models (orthogonal and symplectic) were studied in the double scaling limit, and these gave rise to continuum models with contributions from non-orientable worldsheets. The physics is then encoded in two functions. The oriented contributions come from u as before, which (for the $m = 2$ case) satisfies equation (4.1); the unoriented contributions come from a new function, g , which satisfies (for this $m = 2$ case) the equation:

$$g^3 + 6gg' + 4g'' - 6gu - 6u' = 0. \quad (4.2)$$

¹Such as quotients by discrete symmetries such as \mathbb{Z}_2 .

Given the solution of (4.1), equation (4.2) yields three possible solutions for the asymptotic expansion of $g(z)$. We have:

$$\begin{aligned} u(z) &= \sqrt{z} - \frac{\nu^2}{24z^2} - \frac{49\nu^4}{1152z^{9/2}} + \cdots, \\ g_1(z) &= \zeta\sqrt{6}z^{1/4} - \frac{\nu}{2z} - \zeta\frac{5\nu^2\sqrt{6}}{24z^{9/4}} + \cdots, \\ g_2(z) &= -\frac{\nu}{2z} - \frac{25\nu^3}{24z^{7/2}} - \frac{15745\nu^5}{1152z^6} + \cdots, \end{aligned} \quad (4.3)$$

where $\zeta = \pm 1$. The free energy of the model is given as the sum of an oriented contribution and an unoriented one:

$$F = F_o + F_u, \quad \text{where} \quad F_o'' = -\frac{1}{2}u, \quad F_u' = -\frac{1}{2}g. \quad (4.4)$$

Half of the oriented theory's free energy makes up the oriented contribution to the free energy of the model. The remainder comes from g . We showed in Chapter 3 how terms in the free energy associated with $u(z)$ could be identified with $g_s^{-\chi}$, where $\chi = 2 - 2h$ is the Euler number on a surface with h handles. By analogy, the non-orientable contributions, F_u , represent surfaces with $\chi = 2 - 2h - c$, where c is the number of crosscaps in the surface. So the first term in $g_1(z)$ is the projective plane, \mathbb{RP}^2 , the second term is the Klein bottle, and so on. The parameter ζ in (4.3) represents that we have a sign choice of the basic cross-cap state [49]. We will denote the appropriate quotient theories as O^+ and O^- depending on the sign of ζ . It turns out that each of the two projections is more naturally associated with either the orthogonal or the symplectic matrix model. The third solution, $g_2(z)$, has no obvious string theory interpretation at the present time; though, since it is a solution of the equations, it seems likely that it will have some string theory significance eventually [45–47].

In Chapter 3 it was explained that the physics of the even m minimal bosonic string theories suffers from a non-perturbative instability. The unoriented theory simply inherits this behaviour. The Type 0A models, defined by (3.67), do not have these problems, and so it is hoped that their unoriented sectors will be non-perturbatively smooth also. To proceed we could try to formulate an appropriate matrix model, and in Section 4.5 we will write down a natural guess for this. However, in [45–48] a surprising structure was discovered, again related to the KdV hierarchy, which served to define the unoriented sectors without reference to the original matrix model. We will proceed by naturally adapting this structure for the Type 0A theories. Before doing this, let us review the results of [42], who investigated the double scaling limits

of self-dual unitary matrix models.

4.2 The Self-Dual Unitary Matrix Model

As explained in Chapter 3, the double scaling limit of the unitary matrix model has recently been given a minimal Type 0B string theory interpretation. This is reviewed in Section C.3. The $\hat{c} = 0$ theory with no background ZZ branes ($\Gamma = 0$) has the following string equation:

$$h^3 - h'' - hz = 0, \quad (4.5)$$

In [42] self-dual unitary matrix models were studied. It was found that there are again two contributions to the partition function, one coming from orientable surfaces, $h(z)$, which satisfies equation (4.5); and the other, unorientable contribution, from a function, $w(z)$, which satisfies the Riccati-type equation:

$$w' = \frac{1}{4}w^2 + z - \frac{3}{2}h^2. \quad (4.6)$$

Again, the free energy splits into two parts:

$$F = F_o + F_u, \quad \text{where} \quad F_o'' = -\frac{1}{2}h^2, \quad F_u' = -\frac{1}{2}w. \quad (4.7)$$

The results of [42] were derived long before (4.5) was interpreted in terms of $\hat{c} = 0$ Type 0B string theory. It is therefore natural to suppose that this is an unoriented string theory based upon some projection of the 0B theory with $\Gamma = 0$ for $\hat{c} = 0$. Half of the free energy of the oriented theory makes up the oriented sector of this new theory, and the rest is made up of unoriented contributions. It is natural to wonder if there exists a similar equation to (4.6), but with Γ non-zero. We will see that this may be the case.

In Section C.3 it is shown that for $\hat{c} = 0$ the 0A and 0B models are interrelated via:

$$u = h^2 + z, \quad (4.8)$$

which ensures that the free energies of the oriented Type 0A and Type 0B sectors are the same in this theory. If we can define unoriented sectors for the Type 0 theories then it is again possible that the 0A and 0B contributions will be related in the pure supergravity case. We will investigate this shortly.

4.3 Lax Operators in Unoriented Minimal String Theory

First, let us digress a short while to reintroduce the elegant pseudodifferential operator language that underpins the KdV hierarchy (c.f. subsection 2.4.1):

$$Q\psi \equiv (\partial^2 - u)\psi = \tilde{\lambda}\psi, \quad (4.9)$$

$$P_m\psi \equiv [Q^{m+1/2}]_+ \psi = \frac{\partial\psi}{\partial t_m}, \quad (4.10)$$

where Q and the P_m are the Lax operators, and $u \equiv u(x, t)$ is a KdV solution². As one would expect, these equations play a key role in the string theory context as well. In [50] the string equation of the bosonic $(2, 2m - 1)$ model was formulated as:

$$[P_{m-1}, Q] = 1, \quad (4.11)$$

where P_m and Q are now defined in terms of $u(z)$ instead of $u(x, t)$. Recall from subsection 2.4.1 that for $m = 2$, P_{m-1} is given by:

$$P_1 = d^3 - \frac{3}{2}ud - \frac{3}{4}u'. \quad (4.12)$$

Equation (4.11) is equivalent to the following ‘Lax’ equations:

$$Q\psi \equiv (d^2 - u)\psi = \lambda\psi, \quad (4.13)$$

$$P_{m-1}\psi = \frac{\partial\psi}{\partial\lambda}, \quad (4.14)$$

since $[\partial/\partial\lambda, \lambda] = 1$.³ It can be shown that equation (4.11) is a consequence of translational invariance of the double scaled matrix model eigenvalue distribution. This reflects the fact that the matrix model is defined on \mathbb{R} , as opposed to the complex matrix model defined on \mathbb{R}^+ to encode the eigenvalue ‘wall’ [38].

In [45–48], the unoriented sector is defined in terms of the above objects as follows. We require that P_{m-1} be factorisable into the following form:

$$P_{m-1} = (d^m + \frac{1}{2}gd^{m-1} + \dots)(d^{m-1} - \frac{1}{2}gd^{m-2} + \dots), \quad (4.15)$$

²Recall that we briefly encountered the first of these equations in a string theory setting as (3.72), in the guise of the FZZT partition function.

³Changing from $\tilde{\lambda}$ to λ for reasons that will become clear in Chapter 5.

In the case of $m = 2$ this yields:

$$P_1 = (d^2 + \frac{1}{2}gd + h)(d - \frac{1}{2}g), \quad (4.16)$$

leading to two equations, which when combined give equation (4.2). We point out here that equation (4.2) can be written as

$$T' + gT = 0, \quad (4.17)$$

where

$$T \equiv g' + \frac{1}{4}g^2 - \frac{3}{2}u. \quad (4.18)$$

A natural sub-family of solutions are those which satisfy $T = 0$, which is the following Riccati-type equation:

$$u = \frac{1}{6}g^2 + \frac{2}{3}g'. \quad (4.19)$$

In fact, as pointed out in [45], studying the equation (4.19) is equivalent to the more restrictive factorisation of P_1 :

$$P_1 = (d + \frac{1}{2}g)d(d - \frac{1}{2}g). \quad (4.20)$$

It turns out that the $g_1(z)$ perturbative expansions from equation (4.3) are the solutions to the $T=0$ equation, whereas $g_2(z)$ corresponds to a solution that satisfies equation (4.17), but not $T=0$.

The same operator language that was used above in the $[P_m, Q] = 1$ bosonic case can also be used to formulate the Type 0A models. This is because, as shown in [37, 38], the differential operator structure that underlies the minimal bosonic string equations also defines the minimal Type 0A string equations as well. This works as follows.

The first derivative of the string equation arises by defining, for a particular m , the operator [37, 51]:

$$\tilde{P}_m = Q_+^{m+\frac{1}{2}} - \frac{1}{2}zd, \quad (4.21)$$

and imposing the fundamental equation:

$$[\tilde{P}_m, Q] = Q. \quad (4.22)$$

This equation is a statement of the scale invariance of the eigenvalue distribution with respect to the eigenvalue ‘wall’. This contrasts with equation (4.11), which we

have seen states translation invariance. As in the bosonic models, we can write the following pair of ‘Lax’ equations:

$$\begin{aligned} Q\psi \equiv (d^2 - u)\psi &= \lambda\psi, \\ \tilde{P}_m\psi &= \lambda \frac{\partial\psi}{\partial\lambda}, \end{aligned} \quad (4.23)$$

since $[\lambda\partial/\partial\lambda, \lambda] = \lambda$.

4.4 Unoriented Type 0A

Now we propose that it is natural to define the unoriented contribution by requiring that \tilde{P}_m factorises in the same way as P_{m-1} . This will define physics for all the different m cases in a manner analogous to the multicritical points of [47], but let us concentrate on the case of $m = 1$ (pure supergravity) to see how things work explicitly.

Before proceeding, we note that the scaling operator defined in equation (4.21) is more restrictive than necessary, and misses potentially important physics. More generally, adding a constant, νC_0 to the operator \tilde{P}_m gives an operator which is just as good as a scaling operator, the equation (4.22) remaining true.

So we have⁴:

$$\tilde{P} = -d^3 + \frac{3}{2}ud + \frac{3}{4}u' - \frac{1}{2}zd + \nu C_0 = -(d^2 + \frac{1}{2}gd + h)(d - \frac{1}{2}g), \quad (4.24)$$

and upon expanding and equating powers of derivatives we find two equations:

$$-g' + h - \frac{1}{4}g^2 = -\frac{3}{2}u + \frac{1}{2}z, \quad \frac{1}{2}hg + \frac{1}{4}gg' + \frac{1}{2}g'' = \frac{3}{4}u' + \nu C_0, \quad (4.25)$$

from which, after elimination of h we obtain:

$$g^3 + 6gg' + 4g'' - 6gu - 6u' + 2zg = 8\nu C_0. \quad (4.26)$$

Given the experience of the previous two cases, the question arises as to whether there is a first order Riccati-type equation to which this is equivalent. Noting that [47] showed that one can restrict to that form by a more specific factorisation of P_{m-1} , we try the same thing for \tilde{P}_m :

$$\tilde{P} = -d^3 + \frac{3}{2}ud + \frac{3}{4}u' - \frac{1}{2}zd + \nu C_0 = -(d + \frac{1}{2}g)d(d - \frac{1}{2}g), \quad (4.27)$$

⁴It turns out that we must choose the negative sign of the square root of Q^3 for consistency with $[\tilde{P}, Q] = Q$.

giving:

$$-\frac{1}{4}g^2 - g' = -\frac{3}{2}u + \frac{1}{2}z, \quad \frac{1}{4}gg' + \frac{1}{2}g'' = \frac{3}{4}u' + \nu C_0 \quad (4.28)$$

In order for these two equations to make sense we require that they must have the same content. To do this we must set $C_0 = -1/4$, which means the equation for g can be written simply as:

$$S = 0, \quad (4.29)$$

where

$$S = g' + \frac{1}{4}g^2 - \frac{3}{2}u + \frac{1}{2}z \equiv T + \frac{1}{2}z. \quad (4.30)$$

Using this notation we find that the result of the less restrictive factorisation (4.26) can be expressed as (again with $C_0 = -1/4$):

$$S' + gS = 0, \quad (4.31)$$

We see that at least some of the possible solutions to this equation can arise from $S = 0$ (4.29). This provides a non-trivial check of our conjecture, since using the map (4.8), together with $z \mapsto -z$ and $g \mapsto w$, the equation $S = 0$ becomes equation (4.6), derived for the self-dual unitary matrix model that we conjecture pertain to the Type 0B system. So we have extended the map between oriented 0A and 0B to their unoriented descendants. This also serves to put our computations on a firm double scaled matrix model footing.

It is interesting to examine the behaviour of the solutions in each asymptotic regime. Recall the solutions of equation (3.67) for $u(z)$ in the large positive z (3.68) and negative z (3.69) directions, respectively:

$$u_+ = z + \frac{\nu\Gamma}{z^{1/2}} - \frac{\nu^2\Gamma^2}{2z^2} + \frac{5}{32} \frac{\nu^3\Gamma(4\Gamma^2 + 1)}{z^{7/2}} + \dots, \quad (4.32)$$

$$u_- = \frac{\nu^2(4\Gamma^2 - 1)}{4z^2} + \frac{\nu^4(4\Gamma^2 - 1)(4\Gamma^2 - 9)}{8z^5} + \dots. \quad (4.33)$$

Recall also that for large positive z the relevant worldsheet diagrams come with a factor $g_s^{2h+b-2}\Gamma^b$, where b is the number of boundaries and h is the number of handles. In the negative z expansion on the other hand, Γ^2 is to be taken as representing a single R-R flux insertion [31], and so the first term comprises the torus and the sphere with a flux insertion.

With each of these asymptotics comes two choices for $g(z)$. We will denote these as $g_1(z)$ and $g_2(z)$. The first of them satisfies $S = 0$; the second is only ever a solution of (4.31). For large positive z we have:

$$g_{+,1}(z) = 2\zeta \sqrt{z} + \frac{\nu (3\zeta \Gamma - 2)}{2z} - \frac{\nu^2 (21\zeta \Gamma^2 - 36 \Gamma + 20\zeta)}{16z^{5/2}} + \dots, \quad (4.34)$$

$$g_{+,2}(z) = -\frac{\nu}{z} + \frac{9\nu^2 \Gamma}{4z^{5/2}} - \frac{15\nu^3 (3\Gamma^2 + 2)}{8z^4} + \dots, \quad (4.35)$$

while for the large negative z regime we have:

$$g_{-,1}(z) = i\zeta \sqrt{2z} - \frac{\nu}{z} - \zeta \frac{i\sqrt{2}\nu^2 (12\Gamma^2 - 13)}{8z^{5/2}} - \frac{3\nu^3 (12\Gamma^2 - 13)}{4z^4} + \dots, \quad (4.36)$$

$$g_{-,2}(z) = -\frac{\nu}{z} - \frac{3\nu^3 (12\Gamma^2 - 13)}{4z^4} - \frac{3\nu^5 (336\Gamma^4 - 3384\Gamma^2 + 3085)}{16z^7} + \dots, \quad (4.37)$$

where $\zeta = \pm 1$. An examination of the positive z regime for the first case (g_1) shows (after integrating once and dividing by ν) an expansion in g_s again. It is clear that the first two terms are the contributions of \mathbb{RP}^2 , the Möbius strip, and the Klein bottle, respectively. The diagrams come with a factor $g_s^{2h+b+c-2}\Gamma^b$ where c is the number of crosscaps. In the negative z regime we have an interpretation with just R–R fluxes again, the first term being \mathbb{RP}^2 and the next the Klein bottle. The next has contributions from closed surfaces with the following: $c = 1, h = 0$ and a flux insertion; $c = 3, h = 0$; $c = 1, h = 1$.

For the second case, $g_2(z)$, there is again a surface interpretation consistent with there being D–branes in the positive z regime, and fluxes in the negative z regime. However, just like for the corresponding solution $g_2(z)$ of the bosonic model (4.3), there are missing orders that deserve some explanation, which is perhaps to be found in a study of the continuum theory⁵. As in the bosonic models, we have a free parameter ζ appearing in the $g_1(z)$ expansion. Just like in the bosonic case this should be interpreted as the basic sign choice when defining the cross-cap state.

Plots of the non-perturbative completions of these two cases are given in Figure 4.1 (c.f. Figure 3.4). As in the case of the $[P, Q] = 1$ formulation, it is tempting to speculate that all of these perturbative expansions for $g(z)$ are on equal footing;

⁵There have been some recent studies of noncritical string theory on unoriented surfaces which may be relevant [17, 49, 52, 53].

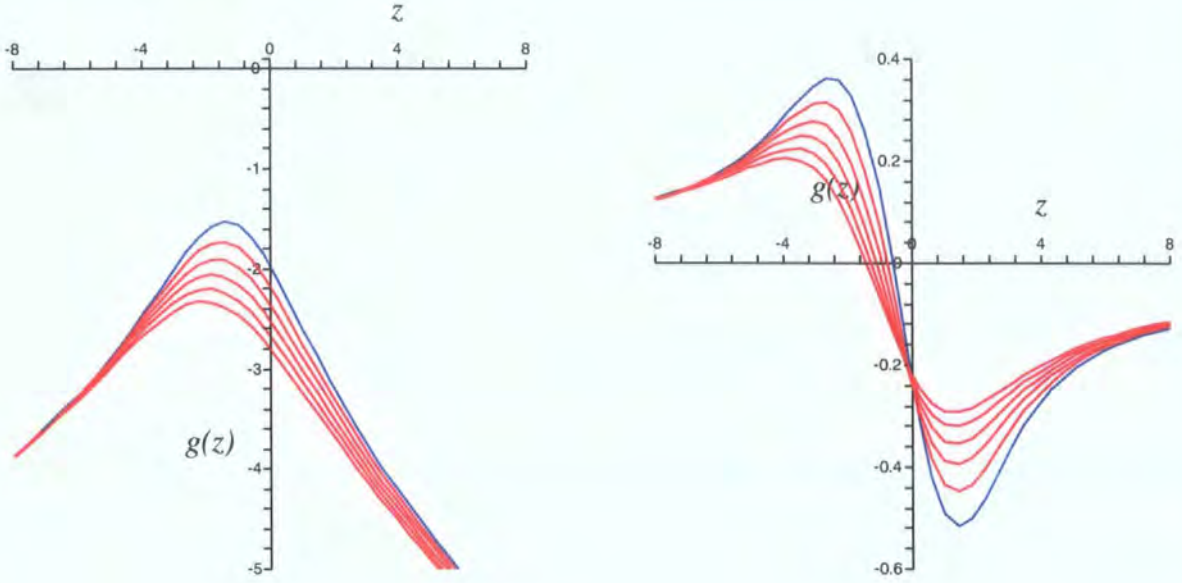


Figure 4.1: Two non-perturbative solutions for the function $g(z)$ which contributes to the unoriented sector of the model. The curves plotted are for $\Gamma = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$, with the blue curves being $\Gamma = 0.0$.

that is, they are equally valid in unorientable string theory.

Having arrived at the case $C_0 = -1/4$, it is natural to wonder whether the other choices for C_0 are physical as well. It would be nice if we had some argument to fix C_0 , possibly in terms of Γ itself. However, there seems to be no obvious way to do this. Therefore, until otherwise proven, we must assume it is a free parameter⁶. We find that we can rewrite equation (4.26) as the following:

$$S' + gS = 2\nu \left(C_0 + \frac{1}{4} \right) \equiv \nu G, \quad (4.38)$$

where we have defined the constant G for later convenience. The case before was $G = 0$. To what do other values of G correspond? Leaving it as a free parameter in

⁶Notice that we can change C_0 in (4.23) to any value we choose by setting $\psi = \lambda^{C_1} \psi_0$, for some C_1 . Recall from Chapter 2 that the Miura map is defined by setting $\psi = e^{-\int v(z) dz}$. We therefore see that C_1 arises from including a $C_1 \ln|\lambda|$ integration constant in this definition.

the equation we find that for large positive z we have:

$$g_{+,1}(z) = 2\zeta\sqrt{z} + \frac{\nu(3\zeta\Gamma - 2 + G)}{2z} - \frac{\nu^2(21\zeta\Gamma^2 + 12\Gamma(G - 3) + 3\zeta G^2 - 18\zeta G + 20\zeta)}{16z^{5/2}} + \dots, \quad (4.39)$$

$$g_{+,2}(z) = -\frac{\nu(G + 1)}{z} + \frac{3\nu^2\Gamma(2G + 3)}{4z^{5/2}} - \frac{\nu^3(3\Gamma^2(8G + 15) + 2G^2(G + 9) + 46G + 30)}{8z^4} + \dots, \quad (4.40)$$

while for the large negative z regime we have:

$$g_{-,1}(z) = i\zeta\sqrt{2z} - \frac{\nu(G + 1)}{z} - \zeta\frac{i\sqrt{2}\nu^2(12\Gamma^2 - 6G^2 - 18G - 13)}{8z^{5/2}} + \dots \quad (4.41)$$

$$g_{-,2}(z) = \frac{\nu(2G - 1)}{z} + \frac{\nu^3(24\Gamma^2G - 36\Gamma^2 - 16G^3 + 72G^2 - 98G + 39)}{4z^4} + \dots \quad (4.42)$$

Figure 4.2 shows some sample plots of the full non-perturbative $g_1(z)$ for varying values of G . Upon examination of the expansions, it is tempting to interpret G as a parameter controlling some aspect of the background, which is present only for the unoriented case. One possibility is that it incorporates the presence of some number, G , of a new type of D-brane that is present only in non-orientable diagrams (since it does not appear in the contributions from oriented surfaces). A candidate such brane could be one which is forced to remain stuck at (the analogue of) orientifold planes, a sort of fractional brane⁷. If this is the case then we would no longer have purely closed strings in the large negative z regime. It is the case $G = 0$ that has a known matrix model interpretation (by virtue of the map to self-dual unitary matrix models discussed earlier), and this would correspond to having such branes absent.

In [2, 44, 49] possible orientifold projections of the Type 0 theories were considered. In the bosonic case we could only quotient the theory by Ω , giving the O^\pm projections. However, in the Type 0 theories we have another choice. We can combine the worldsheet parity operator Ω with the operator $(-1)^{F_L^2}$ (which is also a symmetry of the theories), where F_L^2 is the left-moving part of the spacetime fermion number. This gives a theory quotiented by the operator $\hat{\Omega} = \Omega \cdot (-1)^{F_L^2}$, and corresponds to the \hat{O}^\pm

⁷A discussion of these objects can be found in [1].

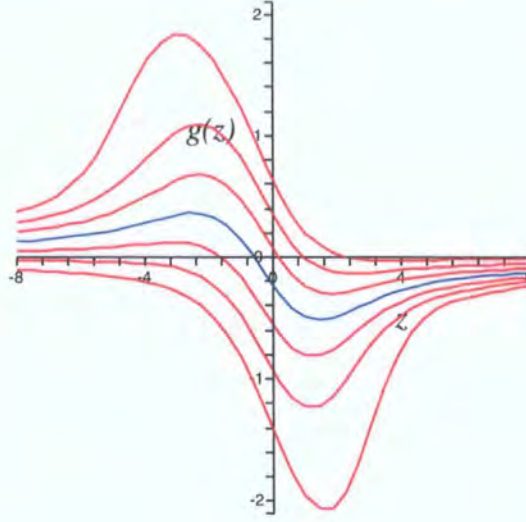


Figure 4.2: The non-perturbative solution for the function $g_2(z)$ which contributes to the unoriented sector of the model. All the curves are for $\Gamma = 0.0$, and from top to bottom we have $G = -0.9, -0.6, -0.3, 0.0, 0.3, 0.6, 0.9$.

projections⁸. It would be insightful to try and tie this in with the above. Perhaps this will help to explain the role of the $g_2(z)$ solution, although why its analogue also appears in the bosonic case would need to be explained.

4.5 A Matrix Model Realisation?

Recall the basic partition function of the Hermitian matrix model in eigenvalue form (3.30):

$$Z_N = \int_{-\infty}^{\infty} \prod_i^N d\lambda_i \Delta^2(\lambda) e^{-\frac{N}{\gamma} V(\lambda_i)}, \quad (4.43)$$

where $\Delta(\lambda)$ is the Vandermonde determinant. It was this partition function that led to the string equation of the oriented bosonic models. As we have already explained,

⁸Actually, in critical Type 0 theories we can also orientifold by the operator $\Omega' = \Omega \cdot (-1)^{F_L}$, where F_L is the left-moving worldsheet fermion number. This is not possible in noncritical string theories because the bulk cosmological constant term in the Liouville action breaks the the global $(-1)^{F_L}$ symmetry.

studies of symmetric matrix models (orthogonal and symplectic) led to the unoriented bosonic models of Section 4.1. In [47] the orthogonal models were studied. These have the following partition function:

$$Z_N = \int_{-\infty}^{\infty} \prod_i^N d\lambda_i \Delta(\lambda) e^{-\frac{N}{\gamma} V(\lambda_i)}, \quad (4.44)$$

which is identical apart from the Vandermonde determinant no longer appears squared. When taking the double scaling limit this makes matters more difficult, and consequently one has to work with the odd eigenvalues, λ_i ($i \in 2\mathbb{Z} + 1$), and the even eigenvalues, λ_i ($i \in 2\mathbb{Z}$), separately. This gives the two functions $u(z)$ and $g(z)$. In [48] symplectic matrix models were studied. These yield the same string equations as the orthogonal models and correspond to:

$$Z_N = \int_{-\infty}^{\infty} \prod_i^N d\lambda_i \Delta^4(\lambda) e^{-\frac{N}{\gamma} V(\lambda_i)}, \quad (4.45)$$

where we see that the Vandermonde determinant now appears raised to the fourth power. It turns out that $\Delta^4(\lambda)$ can be rewritten as:

$$\Delta^4(\lambda) = \prod_{1 \leq i \leq j \leq N} (\lambda_i - \lambda_j)^4 = \det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{2N-1} \\ 0 & 1 & 2\lambda_1 & \dots & (2N-1)\lambda_1^{2N-2} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{2N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2\lambda_N & \dots & (2N-1)\lambda_N^{2N-2} \end{bmatrix}. \quad (4.46)$$

Using elementary properties of determinants this can be rewritten as the determinant of a matrix whose $(2i-1, j)$ entry is $p_{j-1}(\lambda_i)$ and whose $(2i, j)$ entry is $p'_{j-1}(\lambda_i)$, where $p_j(\lambda)$ are the orthogonal polynomials of the matrix model (c.f. Section C.2). If we choose $V(\lambda) = a\lambda^2/2 + b\lambda^4/4$ then, by using the recursion relations for the orthogonal polynomials⁹, integration by parts and basic properties of determinants (decomposition into a sum of minors), one can obtain the following recursion relation for Z_N itself:

$$\begin{aligned} Z_{N+1} &= (N+1)(2N+1)h_{2N}Z_N + 4N^2(N+1)\beta b h_{2N-1}h_{2N+1}Z_{N-1} \\ &\quad - 8N(N^2-1)\beta^3 b^3 h_{2N-1}h_{2N}h_{2N+1}Z_{N-2}, \end{aligned} \quad (4.47)$$

$$\text{with} \quad \int_{-\infty}^{\infty} d\lambda e^{-2\beta V(\lambda)} p_i(\lambda)p_j(\lambda) = h_j \delta_{ij}, \quad \beta = \frac{N}{2\gamma}.$$

⁹In the Hermitian matrix model these are given by (C.2.14).

It is this recursion relation, when double scaled, that yields (4.2). The function $u(z)$ enters via the h_i , and the function $w(z)$ enters in relation to the ratio Z_N/Z_{N-1} . Together they add up to give the total free energy of the model.

Now recall that in moving from the bosonic models to the Type 0A models we moved from the Hermitian matrix model to the rectangular complex matrix model. This had the following partition function (3.66):

$$Z_N = \int_0^\infty \prod_{i=1}^N dy_i y_i^\Gamma \Delta^2(y) e^{-\frac{N}{\gamma} V(y_i)}. \quad (4.48)$$

This is identical to (4.43) except for the factor of y_i^Γ and the fact that the integral is now defined on the half-line rather than the whole real axis. This makes matters more difficult because many of the manipulations involving integration by parts now involve boundary terms, $p_i(0)$, which may be non-zero [33]. This represents the presence of the eigenvalue wall. Of course, the rectangular matrix model is still tractable, and it is the presence of this wall that leads us to (3.67).

We are now in a position to propose a matrix model for unoriented Type 0 string theory. There seem to be two obvious choices:

$$Z_N = \int_0^\infty \prod_{i=1}^N dy_i y_i^\Gamma \Delta(y) e^{-\frac{N}{\gamma} V(y_i)}, \quad (4.49)$$

or

$$Z_N = \int_0^\infty \prod_{i=1}^N dy_i y_i^\Gamma \Delta^4(y) e^{-\frac{N}{\gamma} V(y_i)}, \quad (4.50)$$

which we could tentatively refer to as the orthogonal and symplectic rectangular matrix models respectively (although this is not strictly correct). Both these models correspond to the original orthogonal and symplectic matrix models, but on the half-line and with the additional factor of y_i^Γ corresponding to ‘rectangularity’.

The problem comes when we try to solve either of these models. So far we have made little progress with the first (orthogonal) case. Work on the second case involves formulating a recursion relation analogous to (4.47). However, the boundary conditions at the wall make this intractable. One can successfully generate the recursion relation, but now the terms involve contributions from the $p_i(0)$. Unfortunately, the series for Z_{N+1} appears to contain *all* of the lower Z_M ($M = 1, \dots, N$), and so

involves an infinite number of terms as $N \rightarrow \infty$. Therefore it seems impossible to obtain a sensible relation in the double scaling limit. This is the same problem that was initially thought to afflict the orthogonal matrix model, (4.44), but it was eventually rescued by some devious manipulations [47]. It is possible that similar tricks could be applied to solve (4.49) and (4.50) in the future.

Finally in this section, it would be interesting if we could uncover the role of the parameter G in a matrix model context. This may shed some light on whether or not it really is a free parameter. In the orthogonal rectangular model we would presumably have to deal with odd and even eigenvalues separately again. Since Γ is associated with the even eigenvalues, then perhaps G is somehow associated with their odd counterparts. However, without being able to compute the matrix partition function explicitly, demonstrating this will be difficult.

4.6 Summary

In this chapter we have identified a family of string equations, related to the Lax operators of the KdV hierarchy, that would naturally correspond to unoriented projections of Type 0A string theory. By using the map of (4.8), we tested this conjecture against the results of the double-scaled self-dual unitary matrix model [42], which we speculate corresponds to the $\hat{c} = 0$ Type 0B theory with $\Gamma = 0$. We then attempted to formulate matrix model realisations of unoriented Type 0A and identified two very natural candidates. Unfortunately these models are rendered extremely complicated by the presence of non-zero boundary conditions at the eigenvalue ‘wall’, and no solution has been obtained as yet.

In everything we have studied so far we have been looking at the minimal models purely from the worldsheet point of view, and there has been very little connection made to the target space of the theories. In the next chapter we will address this somewhat by studying the FZZT branes of the oriented Type 0A theory. We will also derive an explicit transformation that changes the parameter Γ by unity when it acts. We will see that this is intimately related to the Bäcklund transformation of the KdV hierarchy. Using this, we will be able to give convincing evidence that Γ must assume integer values, as is consistent with its interpretation as the total background ZZ brane charge.

Chapter 5

More On Minimal Superstring Theories

In this chapter, we will further elaborate upon the connection between minimal Type 0 superstring theories and their underlying integrable hierarchy. We will see that many of the results expounded in Chapter 2 will reappear in new guises within string theory. Most important of these will be the Bäcklund transformation of (2.58). This will be used to give conclusive evidence that the parameter Γ , controlling the number of background ZZ branes in the string theory (3.67), really is quantised. Then, later in the chapter we will investigate how the FZZT branes appear in the formalism. For zero boundary cosmological constant we will see that these branes are intimately related to the Bäcklund transformation. We will also see that we can in principle understand the effects of a non-zero boundary cosmological constant via a simple set of rules. We will make partial progress with developing these.

5.1 Bäcklund Transformations and The Quantisation of Γ

Recall from the rectangular complex matrix model, (3.66), that Γ arises as the degree of ‘rectangularity’ of the matrix. Therefore this is a good indication that $\Gamma \in \mathbb{Z}$. However, this is not conclusive, because, as we have remarked above, Γ can also be introduced via a number of different methods. For example, it can be introduced by

a simple change of measure in the matrix model¹, or as an integration constant² in various derivations.

Certainly the quantisation of Γ is not at all clear from the string equation itself. In fact, it is known perturbatively that the solutions of the equation have special properties for various fractional values of Γ .³ These might well turn out to be unphysical values, but this is not *a priori* clear. So it would be nice to find a clear argument for the quantisation of Γ purely from the integrable system. This aim is most strongly expressed in Klebanov, Maldacena, Seiberg's original paper [31], which first identified the Type 0A and Type 0B string equations.

In the next subsection we will examine the numerical solutions to the string equation in more detail. This will lead us to some interesting observations.

5.1.1 Numerical Solutions of The String Equation

For clarity we once again give the basic string equation (3.67):

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2, \quad (5.1)$$

which was underpinned by the following Schrödinger equation (4.23):

$$Q\psi \equiv (d^2 - u)\psi = \lambda\psi, \quad (5.2)$$

and has the following large z asymptotics (3.68, 3.69):

$$u = z + \frac{\nu\Gamma}{z^{1/2}} - \frac{\nu^2\Gamma^2}{2z^2} + \frac{5}{32} \frac{\nu^3\Gamma(4\Gamma^2 + 1)}{z^{7/2}} + \cdots \quad \text{as } z \rightarrow +\infty, \quad (5.3)$$

$$u = \frac{\nu^2(4\Gamma^2 - 1)}{4z^2} + \frac{\nu^4(4\Gamma^2 - 1)(4\Gamma^2 - 9)}{8z^5} + \cdots \quad \text{as } z \rightarrow -\infty, \quad (5.4)$$

In Chapter 2 we saw how the spectrum of the wavefunction, ψ , contains important information that can be used to access to the non-perturbative physics of the model. Then, in Section 3.8 we saw that λ arises in the bosonic theories as the remnant of

¹Including a logarithmic term in the matrix model potential has the effect of inserting holes of all sizes into the string worldsheets [54–57].

²Note that in the Type 0B theories of Section C.3, the *only* known way to obtain Γ is as an integration constant.

³Observe in particular the case of $\Gamma = \pm\frac{1}{2}$ in the large negative z regime (3.69). There are several other interesting cases of this type, including a family of exact double pole solutions at half-integer Γ . See [56, 57].

the double-scaled eigenvalue distribution of the Hermitian matrix model. The same interpretation is also true in the complex matrix model, although now we have the eigenvalue ‘wall’ at $\lambda = 0$, meaning that the Dyson gas has support on $\lambda \leq 0$ [31, 32]. In Chapter 2 we saw that $\lambda > 0$ corresponds to bound states, whereas $\lambda < 0$ corresponds to oscillatory-wave solutions⁴. The presence of the ‘wall’, and the fact that the eigenvalues of the original matrix model form a continuum, therefore suggests that the Schrödinger problem with potential $u(z)$ should have no discrete bound states. We will test this for our numerical solutions in Section 5.1.4 below, where we will examine the properties of the wavefunction ψ in much more detail. For now let us concentrate mainly on the function $u(z)$ itself.

Recall that when obtaining the boundary conditions in (5.3) above, we were forced to take the square-root of Γ^2 . It was explained in Chapter 3 that, although we have chosen the positive sign of the square root, there is also a second solution corresponding to the negative sign choice. Let us temporarily denote these by $u_{+,\Gamma}(z)$ and $u_{-,\Gamma}(z)$. Since Γ represents the background ZZ brane charge we expect the physics to be invariant under $\Gamma \mapsto -\Gamma$. Indeed, the string equation itself obeys this condition, and we can achieve this automatically at the level of the solutions by choosing the solution $u_{+,\Gamma}(z)$ corresponding to positive Γ , and $u_{-,\Gamma}(z)$ corresponding to negative Γ (or vice-versa). This is because $u_{+,\Gamma} = u_{-,-\Gamma}$. We will show below that for integer values of Γ the solutions actually satisfy the stronger conditions $u_{+,\Gamma} = u_{+,-\Gamma}$ and $u_{-,\Gamma} = u_{-,-\Gamma}$, despite the boundary conditions at large positive z being apparently different. This will be strong evidence that something special is occurring at integer Γ . However, having introduced the notation of u_{\pm} , let us drop it and return to our earlier $u(z) \equiv u_{+,\Gamma}$ with boundary conditions given by (5.3). As we did before, and always bearing in mind the discussion immediately above, we will artificially *define* positive and negative Γ via this expansion for ease of reference.

Numerical solutions to the string equation, (5.1), have already been presented as Figure 3.4 in Chapter 3. These were generated using the equation-solving routine `dsolve` in the package MAPLE 9. Notice that the solutions for positive Γ have a very different character to those at negative Γ . Particularly, in the latter we have the appearance of a prominent potential well. This could potentially lead to the

⁴At least for asymptotically zero potentials, such as $u(z)$ at large negative z .

emergence of bound states of the wavefunction, ψ , in (5.2).

Let us proceed by analysing in more detail the apparent disparity between the two signs of Γ . Starting with the positive case, we find no problems with the numerical computation of the solution for any value of Γ we choose⁵. Moving on to negative Γ , our first observation is that solving the differential equation numerically became much more prone to error, and we had to use more accurate methods. We used the NAG library of routines for solving boundary value problems with C++ to make further progress, as they gave much greater control over the problem. In this way we were able to directly construct a solution for $u(z)$ for values as low as $\Gamma = -0.97$ before running into numerical problems. See Figure 5.1. Notice that the rate of growth of the

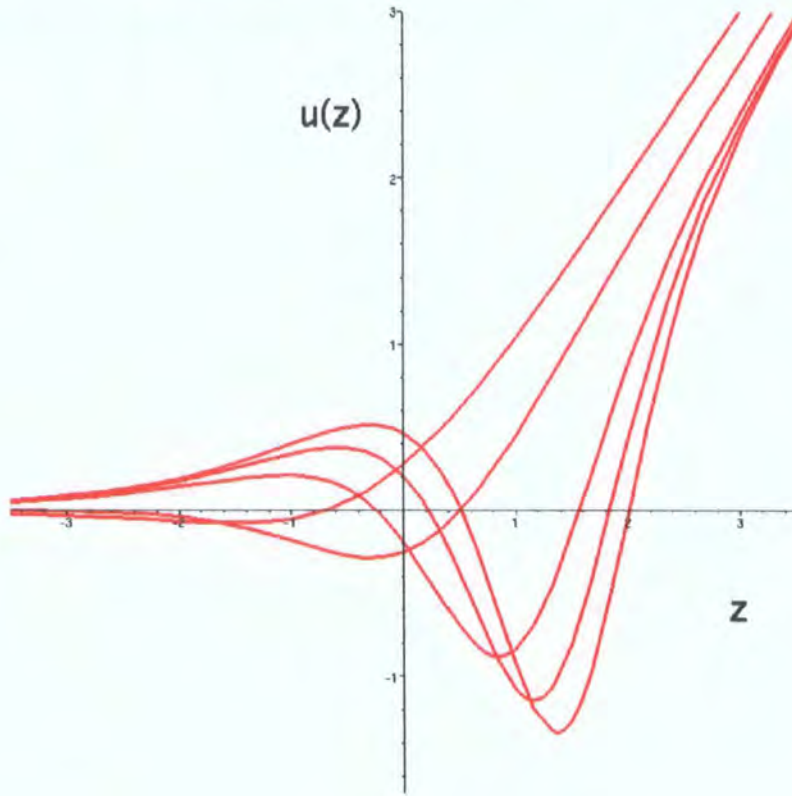


Figure 5.1: Numerical solutions to equation (5.1) in the case $m = 1$, for a range of negative values of the parameter Γ . From left to right we have $u(z)$ for $\Gamma = 0.00, -0.50, -0.90, -0.95, -0.97$, with the well deepening as we approach $\Gamma = -1$.

potential well gets significantly faster as we get closer to $\Gamma = -1$. This, combined with

⁵Though it is obvious that the larger Γ becomes, the more terms in the boundary conditions (5.3, 5.4) we must retain.

the growing instability, is suggestive that something interesting might be happening in this limit. Specifically, it looks as if MAPLE is having problems finding a solution with $u(z) \sim z - \nu/z^{1/2}$ large positive z asymptotics. In order to establish this more satisfyingly, we employed some exact methods (in the form of a solution-generating transformation), which we will describe next. This will allow us to generate new $u(z)$ from the ones we can already extract numerically, and allow us to approach $\Gamma = -1$ in a controlled manner, and with convincing accuracy.

5.1.2 Bäcklund Transformations

Recall from Chapter 2 that solutions to the KdV hierarchy are related to those of the mKdV hierarchy by the Miura map $u = v^2 - v$. Since this is derived from the wave equation, (5.2), one might suspect that the map will also be relevant in string theory. For instance, in the case of the $(2, 4)$ model with $\mathcal{R} = u - z$, the string equation (5.1) is related, via the Miura map $u = v^2 - v$, to the Painlevé II equation:

$$\frac{1}{2}v'' - v^3 + zv + \nu C = 0, \quad \text{where } C = \frac{1}{2} \pm \Gamma. \quad (5.5)$$

Here the sign choice in C reflects the fact that for each solution of (5.1), with the asymptotics in (5.3, 5.4), there are *two* solutions of the Painlevé II equations related to it by the Miura map. In [56], similar relationships were developed for the general $(2, 4m)$ series of string equations, and an explicit transformation was derived that changes C by an integer. In Section D.1 we will review the calculations that lead to this transformation. In the same section we will extend their results to derive an explicit transformation that changes Γ by an integer in the string equation for $u(z)$. Here we state only the results because the calculations are quite involved. Firstly, from [56] we find:

$$v_{C\pm 1} = -v_C + \frac{\nu(2C \pm 1)}{\mathcal{R}[v_C^2 \pm v'_C]}. \quad (5.6)$$

This then leads us to the following explicit transformation for $u_\Gamma(z)$:

$$u_{\Gamma\pm 1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\Gamma\mathcal{R}' + 4\nu^2\Gamma^2}{4\mathcal{R}^2}, \quad (5.7)$$

where $\mathcal{R} \equiv \mathcal{R}(u_\Gamma)$.

Recall from Section 2.5 that the KdV auto-Bäcklund transformation was derived in a similar manner. That is, by starting from the Miura map and using the sign flip

symmetry of v in the mKdV equation. It is natural to suggest that this is just the string equation analogue of that Bäcklund transformation. Strictly speaking, it looks like this is not quite what we have here since the string equation (5.1) is displayed with Γ appearing explicitly, so u_Γ and $u_{\Gamma\pm 1}$ are solutions of *different* equations. However, the once-differentiated string equation (which is in fact the one which appears naturally in many derivations; see the beginning of the previous section, and see below), which is

$$\frac{1}{2}\mathcal{R}''' - 2u\mathcal{R}' - u'\mathcal{R} = 0, \quad (5.8)$$

does not have an explicit appearance of Γ .

So we have a genuine Bäcklund transformation for our system. Let us make explicit the connection with the KdV Bäcklund transformation of (2.58). We start with the KdV equations (3.59), which we rewrite as:

$$\alpha_m \frac{\partial \tilde{u}}{\partial t_k} = R'_{m+1}[\tilde{u}] = \left(\frac{1}{4} \partial^3 - \tilde{u} \partial - \frac{1}{2} \frac{\partial \tilde{u}}{\partial x} \right) R_m[\tilde{u}], \quad (5.9)$$

We can search for solutions of the form $\tilde{u}(x, t_m) = t_m^{2\beta_m} u(z)$, where $z = x t_m^{\beta_m} \nu \equiv \tilde{z} \nu$.⁶ We then find that $\beta_m = -1/(2m+1)$; and, for an appropriate choice of α_m , equation (5.9) becomes:

$$-u - \frac{1}{2} \tilde{z} u' = \left(\frac{1}{4} d^3 - u d - \frac{1}{2} u' \right) R_m[u], \quad (5.10)$$

where a prime denotes the action of $d \equiv \nu d/dz$ as usual. Rearranging, we find the once-differentiated string equation (5.8) for a particular m .

This is another way of making the observation, already noted in [38], that the string equation follows from a restriction of the KdV hierarchy to scale invariant solutions. The derivation leading to this original observation is summarised in Section D.2.⁷ Here, using the map $\tilde{u}(x, t_m) = t_m^{2\beta_m} u(x t_m^{\beta_m} \nu)$, we have made the connection between the string equation and the underlying hierarchy a little more explicit. This also fits with the recovery of the string equation from the operator relation $[\tilde{P}, Q] = Q$ (see (4.22)), because we have seen that \tilde{P} is the generator of scale transformations. Let

⁶More generally we search for solutions of the form $\tilde{u}(x, t_m) = t_m^{\gamma_m} u(z)$; but we find that consistency forces us to choose $\gamma_m = 2\beta_m$.

⁷The formalism described in Section D.2 allows any arbitrary combination of the \tilde{t}_k in (3.56) to be switched on at once. We will see in Chapter 6 how we can extend the scaling argument presented in this section to allow us to turn on arbitrary combinations of the \tilde{t}_k .

us note for completeness that we can use same scaling for the KdV Lax operators Q and P_m (see (2.48)) to obtain the operators Q and \tilde{P}_m that are relevant in the string theory. We can obtain (4.23) by simply setting $\psi(x, t) = \psi(z, \lambda)$ in (2.48), where $\lambda = \tilde{\lambda}t^{-2\beta_m}$. By instead setting $\psi(x, t) = t_m^C \psi(z, t)$, where C is a constant, we can also obtain the additive constant in \tilde{P}_m that was discussed in Chapter 4.

Now that we can explicitly relate the string equation to the KdV equation, we can use this knowledge to apply results from the integrable systems literature developed in Chapter 2 directly into the string theory context. Let us start with the KdV auto-Bäcklund transformation (2.58) relating a solution $u_1(x, t)$ to a solution $u_2(x, t)$:

$$w_{1,x} + w_{2,x} = \frac{1}{2}(w_1 - w_2)^2 + 2\tilde{\lambda}, \quad (5.11)$$

where $w_{i,x}(x, t) = u_i(x, t)$, ($i = 1, 2$).

This can be specialised to solutions of the string equation in the same manner as above by writing $u_i(x, t) = t^{2\beta_m} u_i(xt^{\beta_m})$, which gives:

$$w_i(x, t) = t^{\beta_m} w_i(xt^{\beta_m}), \quad (5.12)$$

We therefore obtain:

$$w'_1 + w'_2 = \frac{1}{2}(w_1 - w_2)^2 + 2\tilde{\lambda}t^{-2\beta_m}. \quad (5.13)$$

So we see that for consistency we must set $\tilde{\lambda} = 0$, which has an interesting interpretation to be discussed below. Hence we have

$$w'_1 + w'_2 = \frac{1}{2}(w_1 - w_2)^2. \quad (5.14)$$

So far, it is not clear that this equation changes Γ . In order to establish the connection between this transformation and the one displayed in equation (5.7), in which Γ appears explicitly, we first work perturbatively. Starting with the asymptotic expansions (5.3, 5.4), for (say) u_1 , it is easy to show using equation (5.14) that the asymptotic expansions obtained for u_2 are of the same form, except that Γ has been replaced by $\Gamma \pm 1$. We then expect that they are equivalent non-perturbatively, and have checked that this is the case by working numerically on some explicit solutions. Actually though, matters are more subtle than this. Since (5.14) is a first order differential equation then one would expect an arbitrary constant to be introduced into the Bäcklund transformed function. If we Bäcklund transform the negative z expansion then, although we do have $\Gamma \mapsto \Gamma \pm 1$, we can also pick up an arbitrary constant that

appears throughout the expansion. However, this constant is not compatible with the original string equation and so must be set to zero. We might conclude therefore that the Bäcklund transformation can give us new functions that are not solutions of the string equation. However, this is premature because we have yet to study the role of the time-part of the Bäcklund transformation, (2.61), in the string theory. After scaling it appropriately, we find that it becomes:

$$2(w_1 - w_2) + 2z(w_1 - w_2)' = 3(w_1 - w_2)'(w_1 + w_2)' - (w_1 - w_2)'''. \quad (5.15)$$

Demanding compatibility of both (5.14) and (5.15) does not permit the arbitrary perturbative constant at large negative z .

So we have established that the Bäcklund transformation takes us between solutions with asymptotics given in (5.3, 5.4) that are for Γ s differing from each other by unity. To do this we had to set $\tilde{\lambda} = 0$. Recall that $\tilde{\lambda}$ was just the (negative) energy eigenvalue of the KdV Schrödinger equation (2.33), which we have seen above naturally translates as $\lambda = \tilde{\lambda}t^{-2\beta_m}$ in string theory terms (5.2). We saw in Chapter 2 that the Bäcklund transformation increases or decreases the soliton number of a given solution by unity. We also saw that $\tilde{\lambda}$ controls the speed of these solitons. Accordingly, we deduce that changing Γ by unity also changes the number of formal *zero-velocity* solitons by unity. We use the term “formal” here because $u(z)$ does not have standard soliton boundary conditions (it does not vanish in both asymptotic directions), and because the wavefunctions to which the solitons correspond (via inverse scattering theory) are of zero energy⁸.

Now that we have a method for generating new solutions $u(z)$ starting from old ones, we will return to our numerical study and see if we can better understand what is happening at $\Gamma = -1$.

5.1.3 The Case of $\Gamma = -1$

We established in the last subsection that given a solution u_Γ , we can generate a solution $u_{\Gamma\pm 1}$. In the subsection before, we reported that it was difficult to solve the string equation numerically as Γ became more negative. Numerical precision was lost

⁸We will see in Section 5.1.4 that they are not really bound states either.

rapidly as Γ went below the value -0.9 , whilst positive Γ is very much under control. We can use our transformation from the previous section to surmount this obstacle, by simply solving the equation numerically for $u(z)$ for $\Gamma = \varepsilon$, for ε small and positive, and then use our transformation (5.7) to build a solution for $u(z)$ with $\Gamma = -1 + \varepsilon$. We can simply follow ε to vanishingly small values and therefore learn the properties of $u(z)$ on the approach to $\Gamma = -1$.

We carried this out with very interesting results. See Figure 5.2 for examples. We were able to use this method to generate $u(z)$ for $\Gamma = -1 + \varepsilon$ where ε could easily be taken as small as $\sim 10^{-9}$. The potential well is observed to get more deep and narrow increasingly rapidly as $\varepsilon \rightarrow 0$. In fact, we observe numerically that the well runs to infinity along the line $u(z) = -z$, becoming infinitely deep and narrow in the limit $\Gamma = -1$. It is interesting to note that, since the deepening of the well is accompanied

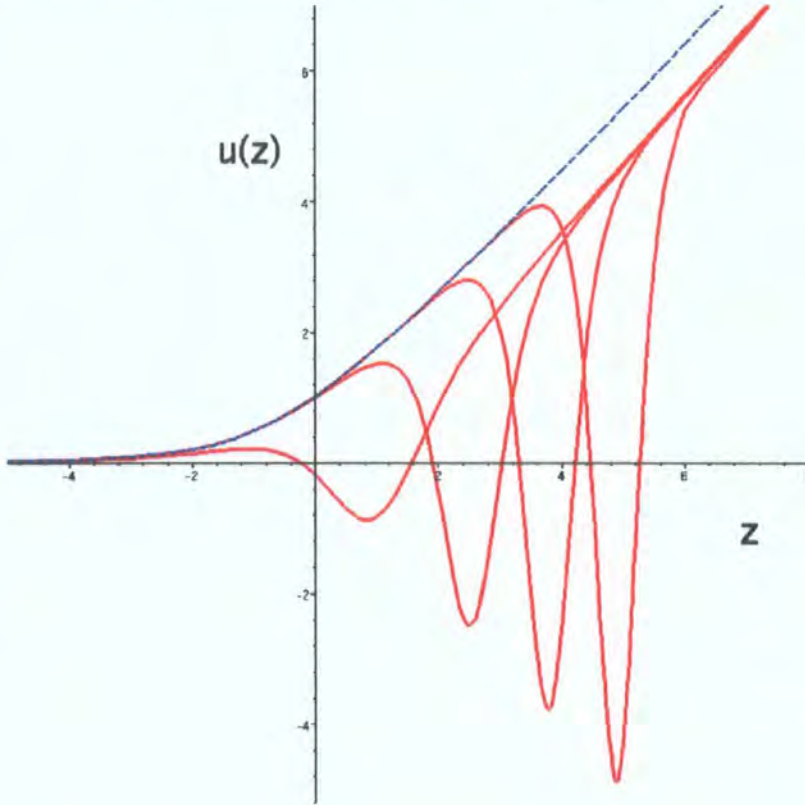


Figure 5.2: Numerical solutions to equation (5.1) in the case $m = 1$, generated using (5.7). The red set of curves show $u(z)$ for $\Gamma = -1 + \varepsilon$, with $\varepsilon = 10^{-2k}$, $k = 1, \dots, 4$ (rightmost curve is $k = 4$). The dashed blue curve is the case of $\Gamma = +1$, used for comparison in the text.

by a narrowing, the system has a chance of preventing the appearance of a bound

state⁹. We will give convincing evidence for this in Section 5.1.4.

So the case $\Gamma = -1$ is quite special, and in fact contains a surprise. At first, it seems that it develops a pathology, since as $\varepsilon \rightarrow 0$ the well was seen to grow infinitely deep, but at the same time it became extremely narrow and, moreover, it moved off towards $z \rightarrow \infty$. So the solution may in fact be well-behaved. In support of this, it is interesting to note (again, from studying families of curves such as those displayed in Figure 5.2) that, in the limit, the $\Gamma = -1$ curve looks rather like the $\Gamma = +1$ curve. The feature associated to the well is confined to a highly localised narrow (and deep) region, which has moved away. Everything that remains of the $\Gamma = -1$ curve at finite z falls (with considerable numerical accuracy) on the $\Gamma = +1$ curve.

This numerical observation suggests the result that the functions $u_{\Gamma=-1}$ and $u_{\Gamma=+1}$ are actually identical, and that the discrepancy from the piece at infinity actually disappears in the limit. This seems to be the case for all the string equations in the hierarchy. Having said that though, it is still an important question is whether the limit $\Gamma \rightarrow -1$ actually exists as defined by the boundary conditions (5.3, 5.4). It seems that the non-perturbative effects at large positive z get stronger the closer Γ gets to -1 , and in the limit the perturbative series is no longer valid at all. Figure 5.2 is very suggestive of this. In fact, the same phenomenon occurs in solutions of Bessel's equation:

$$x^2 \frac{d^2 \phi(x)}{dx^2} + x \frac{d\phi(x)}{dx} + (x^2 - n^2) \phi(x) = 0, \quad (5.16)$$

Here n is an integer that is very reminiscent of Γ in the string equation (we will see later that this relation is even closer). Note in particular that the equation is symmetric under $n \mapsto -n$. Bessel's equation has a basis of two solutions as is to be expected for a second order differential equation [58]. These are labelled by $J_n(x)$ and $J_{-n}(x)$ ($n \notin \mathbb{Z}$), the latter being singular at $x = 0$. These are Bessel functions of the first kind and some examples are shown in Figure 5.3(1).

When $n \in \mathbb{Z}$ one finds the two Bessel functions of the first kind are no longer independent, $J_n = (-1)^n J_{-n}$. Instead the second linearly independent solution is provided by a Bessel function of the second kind, $Y_n(x)$. The key point is that, for

⁹One can obtain a rough estimate of the number of bound states by comparing the depth of the well with the value of the lowest energy eigenstate of the equivalent harmonic oscillator, determined by reading off the numerically calculated value of the second derivative at the bottom of the well. The well always seems to fall short of a bound state, at least down to $\Gamma = -0.970$.

$n \notin \mathbb{Z}$, J_{-n} is singular at $x = 0$, but for $n \in \mathbb{Z}$ it is non-singular. In the same way that we studied the behaviour of $u(z)$ as $\Gamma \rightarrow -1$, we can also study the behaviour of Bessel functions in the limit $n \rightarrow n_0 \in \mathbb{Z}$. This is done using MAPLE's library Bessel function and is shown in Figure 5.3(r). Notice that as $n \rightarrow n_0$ J_{-n} gets closer and

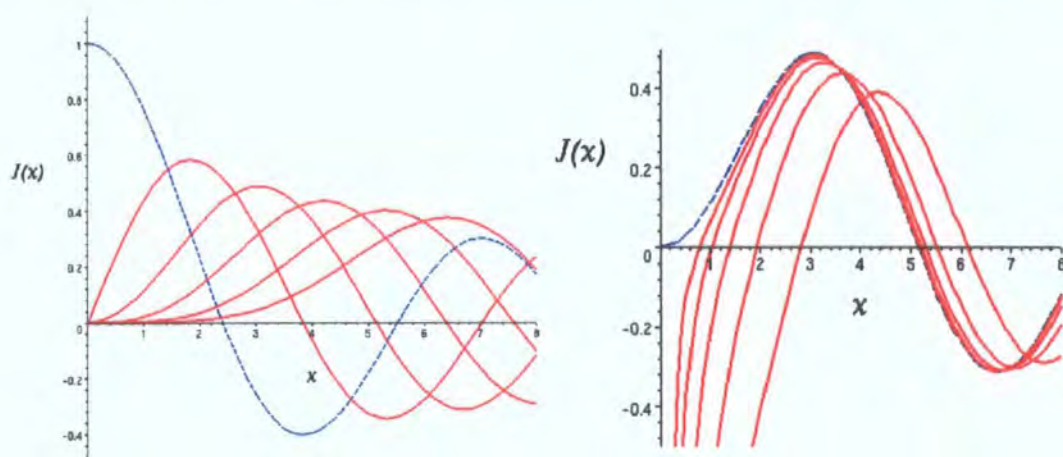


Figure 5.3: (l) Various Bessel functions of the first kind, the dashed blue curve being $J_0(x)$; (r) $J_2(x)$ (dashed blue) is compared to $J_{-2+\epsilon}$ for $\epsilon = e^{-k}/2$, $k = 0, \dots, 4$ (rightmost curve is $k = 0$).

closer to J_{n_0} in all but a vanishingly small region in the neighbourhood of $x = 0$. It is clear that this is the same phenomenon that we have seen with solutions, $u(z)$, to the string equation. So, just as with Bessel functions, we see that we are justified in saying $u_\Gamma = u_{-\Gamma}$ for $\Gamma \in \mathbb{Z}$.

We can understand this using the Bäcklund transformations as well. Starting with $u_{\Gamma=0}$ let us apply the transformation (5.7). This again gives the result that $u_{\Gamma=-1} = u_{\Gamma=+1}$, since there is no explicit appearance of Γ in the transformation in order to generate the difference between decreasing *vs* increasing Γ . In the case of (5.14) and (5.15), analysing perturbatively we seem to get a simple choice of $\Gamma = \pm 1$. However, since the perturbative expansion for $\Gamma = 0$ at large positive z is $u_1(z) = z$ to *all orders in perturbation theory*, we can try to solve the Bäcklund transformation explicitly for $w_2(z)$ (and hence $u_2(z)$) in this regime. Using MAPLE, we find the solution is given in terms of Airy functions, $Ai(z)$ and $Bi(z)$ (ignoring powers of ν for

clarity):

$$w_2(z) = \frac{z^2[Ai(z) + \alpha Bi(z)] - 4[Ai'(z) + \alpha Bi'(z)]}{2[Ai(z) + \alpha Bi(z)]}, \quad (5.17)$$

where α is an arbitrary constant. Actually, putting this solution back into the string equation, (3.67), we find that it is an exact solution for all values of α providing that $\Gamma^2 = 1$. This is because $u(z) = z$ was an exact solution of the bosonic string equation, $\mathcal{R} = 0$, to begin with¹⁰. If we plot the solution for all z then we find that it appears to have an infinite number of double poles on the negative z axis. This is the same problem that scuppers the bosonic theories non-perturbatively, and is linked to the existence of a free non-perturbative parameter [21]. However, in the Type 0A case we are only interested in the solution at large z . We can expand $u_2(z)$ as a series using MAPLE, but we find that instanton effects associated with $y \equiv e^{-2z^{3/2}/3}$ appear. We have:

$$u_2(z) = \left(\frac{4\alpha^2 - 12\alpha y^2 + y^4}{(y^2 + 2\alpha)^2} \right) z - \left(\frac{24\alpha^3 - 8\alpha^2 y^2 + 4\alpha y^4 - 3y^6}{3(y^2 + 2\alpha)^3} \right) \frac{1}{z^{1/2}} + O(z^{-2}). \quad (5.18)$$

We therefore see that we can effectively set $y = 0$ at large z , because its effects are smaller than any term in the perturbative expansion¹¹. The perturbative expansion becomes:

$$u_2(z) = z + \frac{1}{z^{1/2}} + O(z^{-2}) \quad \alpha = 0, \quad (5.19)$$

$$u_2(z) = z - \frac{1}{z^{1/2}} + O(z^{-2}) \quad \alpha \neq 0. \quad (5.20)$$

The first of these is (5.3) with $\Gamma = +1$; the second with $\Gamma = -1$. So we now see the origin of the sign choice when we perform the Bäcklund transformation. There is actually only one solution to the differential equations, but it can have one of two large positive z boundary conditions. Figure 5.4 shows a plot of $u_2(z)$ as given by (5.18) for various values of α . Notice the appearance of the characteristic well associated with the non-perturbative effects as they increase. This looks very similar to Figure 5.2, and it is interesting that one can smoothly take the limit $\alpha \rightarrow 0$. This again suggests that the corresponding limit displayed in Figure 5.2 also exists. All this again ties

¹⁰So (5.17) is the bosonic analogue of the $\Gamma = 1$ solution.

¹¹We see from this that it may actually be possible to retain the naively non-universal sphere term because it is actually non-analytic.

in with the notion of the Bäcklund transformation being related to soliton solutions. One could possibly think of (5.18) as some sort of kink solution connecting the two perturbative vacua of (5.19) and (5.20). If one instead plots $w_2(z)$ directly, as in Figure 5.5, then the solutions look even more kink-like.

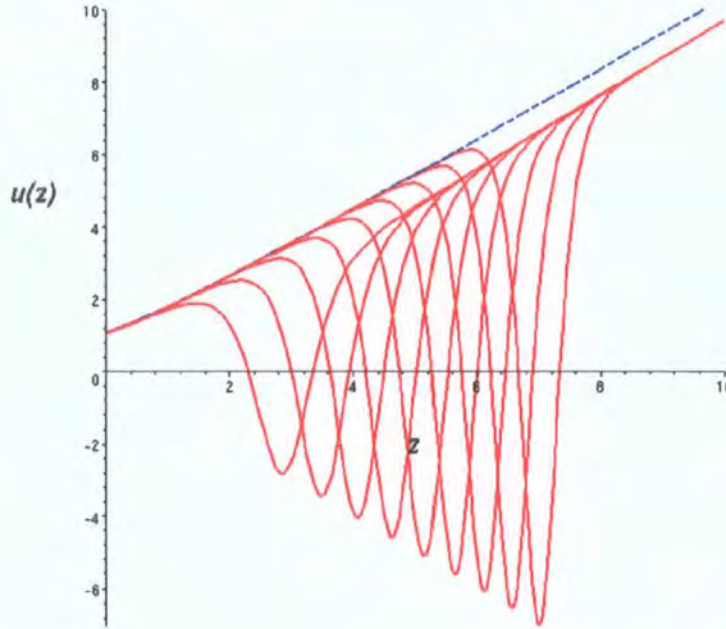


Figure 5.4: The derivative, $u(z)$, of the exact large- z solution, (5.17), to the Bäcklund transformation (starting from $u(z) = z$) is plotted for $\alpha = 10^{-3}, 10^{-4}, \dots, 10^{-11}$ (red curves, left to right). All curves asymptote to the $\Gamma = -1$ boundary conditions. The numerical solution of the string equation for $\Gamma = 1$ is plotted for comparison purposes (dashed blue).

It seems that there are similar non-perturbative parameters analogous to α associated with taking the Bäcklund transformation at other values of Γ . For instance, for non-zero Γ we can leave in up to the next-to-leading order term and we obtain a Bäcklund transformed function related to Whittaker functions. Again there is a non-perturbative parameter that interpolates between the two possible perturbative vacua.

As we have seen above, in the case that we try to match to the boundary conditions of (5.20) for $\Gamma = -1$ we run into problems, and the solution starts to look more and more like (5.19), suggesting that $u_{\Gamma=1} = u_{\Gamma=-1}$. Comparing Figures 5.2 and 5.4,

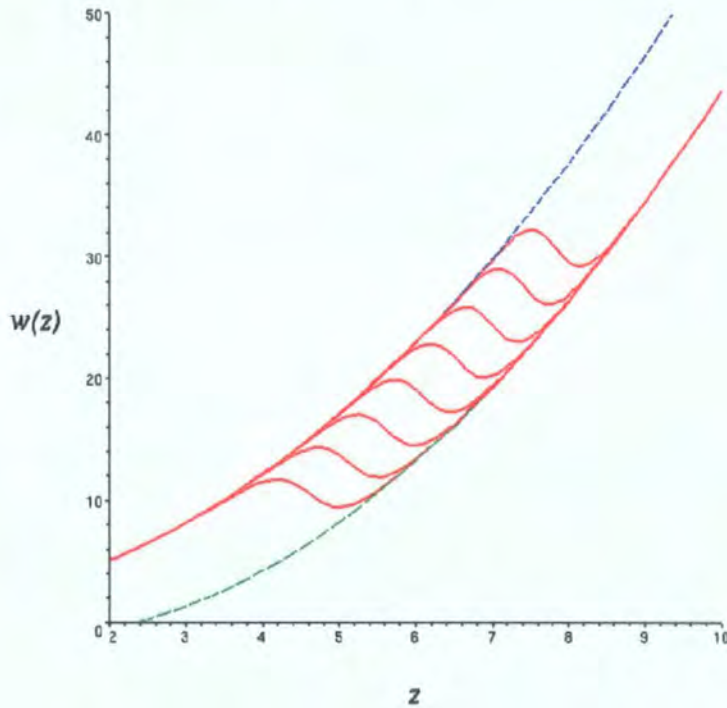


Figure 5.5: The exact large- z solution, (5.17), to the Bäcklund transformation (starting from $u(z) = z$) is plotted for $\alpha = 10^{-6}, 10^{-7}, \dots, 10^{-13}$ (red curves, left to right). All curves interpolate between the $\Gamma = 1$ boundary conditions (dashed blue) and the $\Gamma = -1$ conditions (dashed green).

it is tempting to speculate that perhaps α and Γ are correlated in such a way that at $\Gamma = -1$ the parameter α is forced to be zero. The fixing of α would presumably arise because of the need to match the solution to the large negative z boundary conditions. We have seen that the ‘spatial’ part of the Bäcklund transformation (5.14) of the large negative z asymptotics (3.69) also introduces a constant into the solution, although this time the constant appears perturbatively, not just non-perturbatively. However, this is incompatible with the string equation and the ‘time’ part of the transformation, (5.15) unless the parameter is set to zero. Since the Bäcklund transformation is a first order differential equation, it is clear that there can only be one arbitrary constant in the transformed function. Therefore, the setting of the constant to zero in the negative z regime presumably constrains the positive z constant parameter, α , when matching boundary conditions. This would tie in with what is known about Type 0 solutions of (3.67) because they are thought to be almost-certainly non-perturbatively unique [38] (as is indicated by our numerical results); whereas the bosonic solutions

of $\mathcal{R}[u] = 0$ are known to have non-perturbative ambiguities. Furthermore, in [59] the uniqueness of the $\Gamma = 0$ solution was actually proved, and so presumably all the solutions related to it by the Bäcklund transformation must also share this property.

Note that below $\Gamma = -1$ there is a transition to completely new behaviour. There the system no longer has a smooth solution with the asymptotics given in equation (5.3, 5.4), and there must be a singularity at *finite* z . The origin of the singularity can be seen by considering the form of the transformation (5.7). The denominator of the transformation is $4\mathcal{R}^2$, where, for large positive z , we have that

$$\mathcal{R} = u - z \simeq \frac{\nu\Gamma}{z^{\frac{1}{2}}} + \dots, \quad (5.21)$$

and so, for $\Gamma < 0$, the solution must approach the line $u = z$ from below as $z \rightarrow +\infty$. Since $u(z)$ started out above this line (at $z \rightarrow -\infty$), it must be that it crosses the line somewhere in the interior. If this is the case then, somewhere in the interior, the new solution $u_{\Gamma-1}$ must have a singularity, since the denominator vanishes there. We expect similar arguments to hold for the negative Γ behaviour of the other models in the hierarchy.

So we have the very natural physical situation singling out Γ as a positive integer, since any solution $u(z)$ with $-1 < \Gamma < 0$ can be used as a seed for our transformation (5.7) to generate a $u(z)$ solution with poles, which have a problematic interpretation. Furthermore, any solution $u(z)$ for a positive fractional value of Γ can also be used, by applying the transformation successively, to generate such solutions; so they are in the same class as the $-1 < \Gamma < 0$ solutions. The $u(z)$ for positive integer values of Γ are therefore special, in that they are not connected by our transformation to any solutions with poles.

Recall that the physics should be invariant under charge conjugation symmetry $\Gamma \mapsto -\Gamma$. Although we earlier explained how we could impose this on the solutions ‘by hand’ by an appropriate choice of sign when taking the square root of Γ^2 , it seems non-trivial that the solutions themselves (i.e. with fixed sign choice) are also charge conjugation invariant, but only if $\Gamma \in \mathbb{Z}$. This backs up the evidence that it only integer values of Γ are physically relevant.

Using our experience of the KdV hierarchy, we made the identification that the Bäcklund transformation creates and destroys solitons with zero velocity, at least for-

mally. Recall that in the rectangular matrix model we are dealing with the eigenvalues of the matrix MM^\dagger , where M is an $N \times (N + \Gamma)$ matrix. Using this information it is easy to show that MM^\dagger has $N + \Gamma$ eigenvalues (for Γ positive), but that at least Γ of these eigenvalues are zero. This means that as we change Γ we are adding or subtracting zero eigenvalues. This ties in with the interpretation of the Bäcklund parameter, λ , being related to the double scaled remnants of the eigenvalues: adding extra zero eigenvalues is like adding extra ZZ branes, and this change is mediated by a zero velocity soliton. This connection may have important consequences.

So far we have concentrated on a study of $u(z)$ itself, without explicitly making use of the Baker function $\psi(z, \lambda)$. In the next subsection we will examine the properties of ψ in more detail. This will not only reconfirm what we have already learnt, but will also lay the groundwork for understanding the role of ψ as an FZZT brane partition function.

5.1.4 Properties of The Baker Function

Let us therefore turn to the properties of the wavefunction defined by (5.2). We can make the following observations. The potential becomes linear in the large positive z regime, $u(z) = z + \dots$, and so there the wavefunction will behave like an Airy function. Meanwhile, in the large negative z regime, the potential vanishes to leading order, and at next order is:

$$u(z) = \frac{\nu^2}{z^2} \left(\Gamma^2 - \frac{1}{4} \right) + \dots, \quad (5.22)$$

so the Schrödinger equation is:

$$-\nu^2 \frac{d^2 \psi(z)}{dz^2} + \frac{\nu^2}{z^2} \left(\Gamma^2 - \frac{1}{4} \right) \psi(z) = \lambda \psi(z). \quad (5.23)$$

Notice what happens when we change variables using $\psi(z) = z^{1/2} \phi(z)$, and define $x = \lambda^{1/2} z / \nu$. We get the equation:

$$x^2 \frac{d^2 \phi(x)}{dx^2} + x \frac{d\phi(x)}{dx} + (x^2 - \Gamma^2) \phi(x) = 0, \quad (5.24)$$

which is simply Bessel's equation (5.16) with $n = \Gamma$. This is yet more evidence for the quantisation of Γ . The behaviour (for any Γ) in this regime, after converting back to the original variables, is:

$$\psi(z) = e^{\pm \frac{\lambda^{1/2}}{\nu} z} + \dots. \quad (5.25)$$

Notice that for $\lambda > 0$ the wavefunction will fall (or rise) exponentially as $z \rightarrow -\infty$, indicating that a bound state can possibly form; whereas for $\lambda < 0$ the solution will be oscillatory in this regime. Recall that we expect bound states to be absent because of the eigenvalue ‘wall’ at $\lambda = 0$. It is interesting to establish this absence numerically as the potential well develops for $\Gamma \rightarrow -1$, and we have checked this down to $\Gamma = -0.970$.¹² Even though there are no known bound states, we can still have a continuum of solutions with $\lambda > 0$. These will ‘blow-up’ at either $z \rightarrow \infty$ or $z \rightarrow -\infty$, and so would usually be ruled out in ordinary quantum mechanics. We will see below that solutions of this form play an important role in the string theory.

Let us now focus on the case of vanishing energy ($\lambda = 0$), which we know is associated with the Bäcklund transformation between u_Γ and $u_{\Gamma \pm 1}$. Recall once again that if we set $\psi' = -v\psi$ in the wave equation then we obtain the Miura map, $u = v^2 - v'$. We already know that the function $v(z)$ satisfies the string equations of the mKdV hierarchy, the lowest of which is the Painlevé II equation (5.5). We earlier explained how, for each solution $u(z)$ there correspond two solutions of the mKdV string equation $v_{C_+}(z)$ and $v_{C_-}(z)$, with $C_\pm = \frac{1}{2} \pm \Gamma$. From now on we will often denote these as $v_\Gamma(z)$ and $\bar{v}_\Gamma(z)$ respectively. Let us solve for v_Γ and \bar{v}_Γ in the case of $m = 1$ pure supergravity:

$$\begin{aligned}
 v_\Gamma &= z^{1/2} + \frac{1}{2} \frac{\nu C_+}{z} - \frac{1}{32} \frac{\nu^2}{z^{5/2}} (12\Gamma^2 + 12\Gamma + 5) + \dots ; & \text{as } z \rightarrow +\infty, \\
 v_\Gamma &= -\frac{\nu C_+}{z} - \frac{1}{8} \frac{\nu^3}{z^4} (4\Gamma^2 - 1)(2\Gamma + 3) + \dots ; & \text{as } z \rightarrow -\infty, \\
 \bar{v}_\Gamma &= -z^{1/2} + \frac{1}{2} \frac{\nu C_-}{z} + \frac{1}{32} \frac{\nu^2}{z^{5/2}} (12\Gamma^2 - 12\Gamma + 5) + \dots ; & \text{as } z \rightarrow +\infty, \\
 \bar{v}_\Gamma &= -\frac{\nu C_-}{z} + \frac{1}{8} \frac{\nu^3}{z^4} (4\Gamma^2 - 1)(2\Gamma - 3) + \dots ; & \text{as } z \rightarrow -\infty.
 \end{aligned} \tag{5.26}$$

Integrating and exponentiating, we find that there are two choices for the Baker

¹²This was done using the TQLI method in C++, which amounts to discretising the problem and then diagonalising a tri-diagonal matrix.

function:

$$\begin{aligned}
\psi_\Gamma &= z^{-\frac{1}{2}(\frac{1}{2}+\Gamma)} e^{-\frac{2}{3}z^{3/2}} + \dots ; & \text{as } z \rightarrow +\infty, \\
\psi_\Gamma &= z^{\frac{1}{2}+\Gamma} + \dots ; & \text{as } z \rightarrow -\infty, \\
\bar{\psi}_\Gamma &= z^{-\frac{1}{2}(\frac{1}{2}-\Gamma)} e^{\frac{2}{3}z^{3/2}} + \dots ; & \text{as } z \rightarrow +\infty, \\
\bar{\psi}_\Gamma &= z^{\frac{1}{2}-\Gamma} + \dots ; & \text{as } z \rightarrow -\infty,
\end{aligned} \tag{5.27}$$

This is appropriate, since the potential $u(z) \sim z$ as $z \rightarrow +\infty$ and so the wavefunction should resemble the exponential tail of the Airy function in that limit, and it does. Meanwhile in the $z \rightarrow -\infty$ limit, where the potential vanishes to leading order, the wavefunction has a purely power law behaviour, entirely appropriate for a zero energy state.

The form of $v_\Gamma(z)$, $\bar{v}_\Gamma(z)$ and the associated wavefunctions $\psi_\Gamma(z)$ and $\bar{\psi}_\Gamma(z)$ can be exhibited numerically, and examples are given in Figures 5.6 (for v_Γ), 5.7 (for \bar{v}_Γ), and 5.8, (for ψ_Γ and $\bar{\psi}_\Gamma$). Note that it becomes progressively more difficult to solve for v_Γ (\bar{v}_Γ) with the given boundary conditions as Γ approaches minus-one (zero). However, we will later define Bäcklund transformations for v_Γ and \bar{v}_Γ that allow us to overcome this difficulty. We have made use of these transformations in Figures 5.6(b) and 5.7(b). Notice that both ψ_Γ and $\bar{\psi}_\Gamma$ ‘blow-up’ at either $z \rightarrow \infty$ or $z \rightarrow -\infty$, and hence are non-normalisable regardless of the value of Γ . Once again, we might be tempted to conclude that these wavefunctions were unphysical. However, we will see in Section 5.2 that they are needed to explain the physics of the Bäcklund transformation.

5.1.5 Supersymmetry and Bäcklund Transformations

In this section we will see how the supersymmetric quantum mechanics associated with the KdV hierarchy applies in the string theory context. The interested reader should recall that we have explored the theory behind this in some detail in Section B.6. However, we repeat the salient details here. To start, we factorise the Hamiltonian, $\mathcal{H} \equiv -Q$, as:

$$\mathcal{H}_\Gamma = \mathcal{A}^\dagger \mathcal{A} = -d^2 + u_\Gamma, \tag{5.28}$$

where $\mathcal{A} = d + v_\Gamma$ and $\mathcal{A}^\dagger = -d + v_\Gamma$. The “superpartner” Hamiltonian can be constructed for $\mathcal{H}_{\Gamma+1}$ by simply reversing the order of the factors, to form:

$$\mathcal{H}_{\Gamma+1} = \mathcal{A} \mathcal{A}^\dagger = -d^2 + u_{\Gamma+1} , \tag{5.29}$$

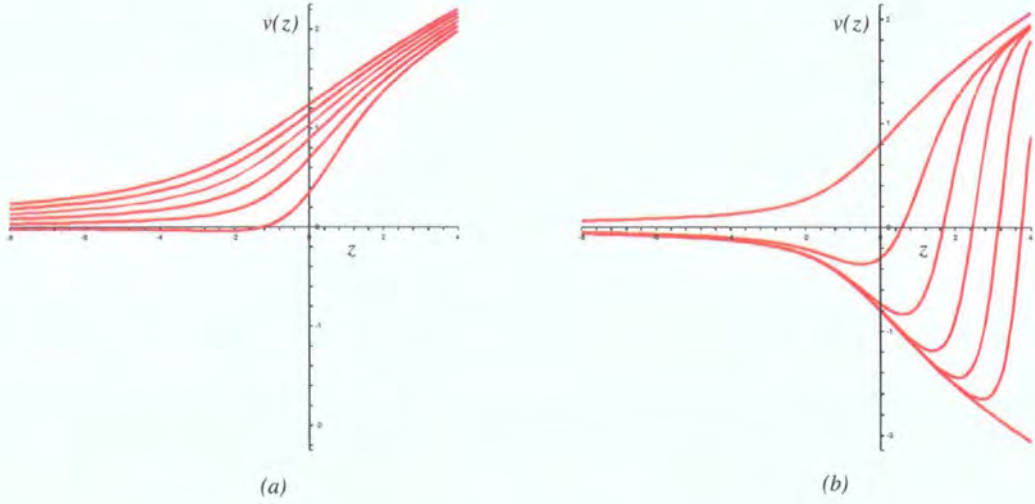


Figure 5.6: Numerical solutions for $v_\Gamma(z)$ for a range of values of Γ . From top to bottom, we have: (a) $\Gamma = 1.4, 1.0, 0.6, 0.2, -0.2, -0.6$; (b) $\Gamma = 0.0, -0.9, -0.99, -0.999, -0.9999, -0.99999$ generated by the Bäcklund transformations (5.53), ending with the curve of $\bar{v}_\Gamma(z)$ for $\Gamma = 1$ (see later in text for explanation).

where we have changed the label on \mathcal{H} to reflect the fact that we now have a new potential, which is:

$$u_{\Gamma+1} = v_\Gamma^2 + v'_\Gamma. \quad (5.30)$$

Since this is just the usual sign-flip on v_Γ , we see that Γ has indeed changed by an integer.

In the same manner we can also construct $\mathcal{H}_{\Gamma-1}$ by instead choosing to do the sign flip on the \bar{v}_Γ function. This gives us the equation $\tilde{C} = -(1 - C) = \frac{1}{2} + (\Gamma - 1)$, telling us that the function $\tilde{u} = \bar{v}_\Gamma^2 + \bar{v}'_\Gamma$ is in fact $u_{\Gamma-1}$, now constructed from the $v_{\Gamma-1}$ function

$$\mathcal{H}_{\Gamma-1} = -d^2 + \bar{v}_\Gamma^2 + \bar{v}_\Gamma. \quad (5.31)$$

In this way we see that we can increase or decrease Γ by an integer depending upon whether we act with the sign flip on v_Γ or \bar{v}_Γ (respectively), and the resulting $u_{\Gamma+1}$ or $u_{\Gamma-1}$ will be constructed from $\bar{v}_{\Gamma+1}$ and $v_{\Gamma-1}$, respectively. The structure is illustrated in Figure 5.9.

We can use this factorisation to determine how the spectra of the Hamiltonians of any two neighbouring Γ are related. If $\psi(z)$ is a wavefunction of \mathcal{H}_Γ with *non-zero* eigenvalue λ , then by left-multiplying the eigenvalue equation on both sides by \mathcal{A} ,

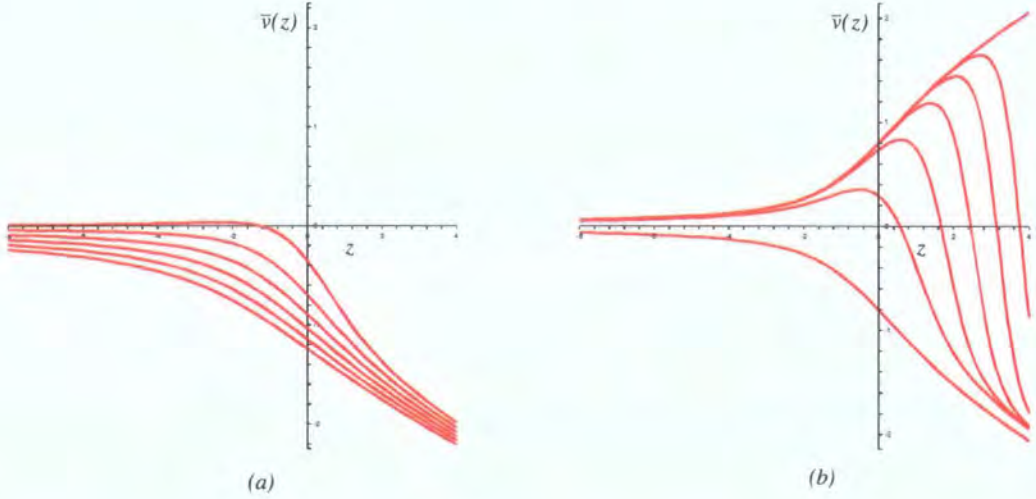


Figure 5.7: Numerical solutions for $\bar{v}_\Gamma(z)$ for a range of values of Γ . From bottom to top, we have: (a) $\Gamma = 2.4, 2.0, 1.6, 1.2, 0.8, 0.4$; (b) $\Gamma = 1.0, 0.1, 0.01, 0.001, 0.0001, 0.00001$ generated by the Bäcklund transformations (5.53), ending with the curve of $v_\Gamma(z)$ for $\Gamma = 0$ (see later in text for explanation).

one can see that it maps under “supersymmetry” to a wavefunction $\mathcal{A}\psi(z)$ of $\mathcal{H}_{\Gamma\pm 1}$ with the *same* energy λ . Away from the zero-energy sector therefore, \mathcal{H}_Γ and $\mathcal{H}_{\Gamma\pm 1}$ have identical spectra. At $\lambda = 0$, the story is slightly different however. There, the map fails, and we find that as we convert the spectrum from \mathcal{H}_Γ to that of $\mathcal{H}_{\Gamma+1}$, any $\lambda = 0$ state is lost. So, once again we see that it is $\lambda = 0$ that is related to changing the number of background ZZ branes.

In the standard nomenclature, we can think of \mathcal{H}_Γ as bosonic and $\mathcal{H}_{\Gamma\pm 1}$ as fermionic, and we have such a pair for any value of Γ . A most efficient way of writing all of this to see the supersymmetric structure is to define the identity matrix, σ_0 , together with the Pauli matrices, σ_j , $j = 1, 2, 3$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.32)$$

(where here i is the square root of -1) and combine \mathcal{H}_Γ and $\mathcal{H}_{\Gamma+1}$ into a larger Hamiltonian H :

$$H = (-d^2 + v^2)\sigma_0 - v'\sigma_3 = \begin{pmatrix} -d^2 + v^2 - v' & 0 \\ 0 & -d^2 + v^2 + v' \end{pmatrix}. \quad (5.33)$$

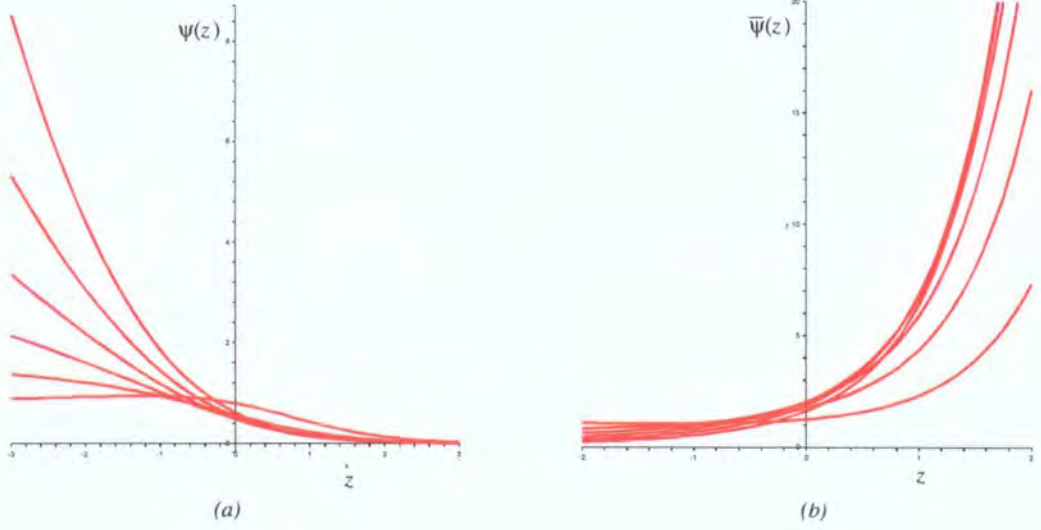


Figure 5.8: Numerical solutions for the wavefunctions $\psi_\Gamma(z)$ and $\bar{\psi}_\Gamma(z)$ for a range of values of Γ : (a) Top to bottom, $\Gamma = 1.4, 1.0, 0.6, 0.2, -0.2, -0.6$; (b) Top to bottom, $\Gamma = 2.4, 2.0, 1.6, 1.2, 0.8, 0.4$

Defining $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, so that $\{\sigma_+, \sigma_-\} = 1$ and $[\sigma_+, \sigma_-] = -\sigma_3$, we can define the supercharges as:

$$Q^\dagger = -i\mathcal{A}^\dagger\sigma_- = i \begin{pmatrix} 0 & 0 \\ -d+v & 0 \end{pmatrix}, \quad Q = -i\mathcal{A}\sigma_+ = i \begin{pmatrix} 0 & d+v \\ 0 & 0 \end{pmatrix}, \quad (5.34)$$

(in the last two displayed equations v can mean v_Γ or \bar{v}_Γ , depending upon whether going from \mathcal{H}_Γ to $\mathcal{H}_{\Gamma+1}$ or to $\mathcal{H}_{\Gamma-1}$) so that we have the supersymmetry algebra:

$$\{Q^\dagger, Q\} = 2H, \quad [Q, H] = 0. \quad (5.35)$$

The condition for supersymmetry for a particular potential $u(z)$ and its superpartner potential is that we have a normalisable zero energy ground state, and we have already seen that for our system's boundary conditions, this fails. Our family of potentials therefore breaks supersymmetry. More details of this phenomenon can be found in Section B.6.

5.2 The FZZT Partition Function

Recall from Chapter 3 that there are two types of brane in minimal string theory: the ZZ branes and the FZZT branes. We have learnt in great detail how the function

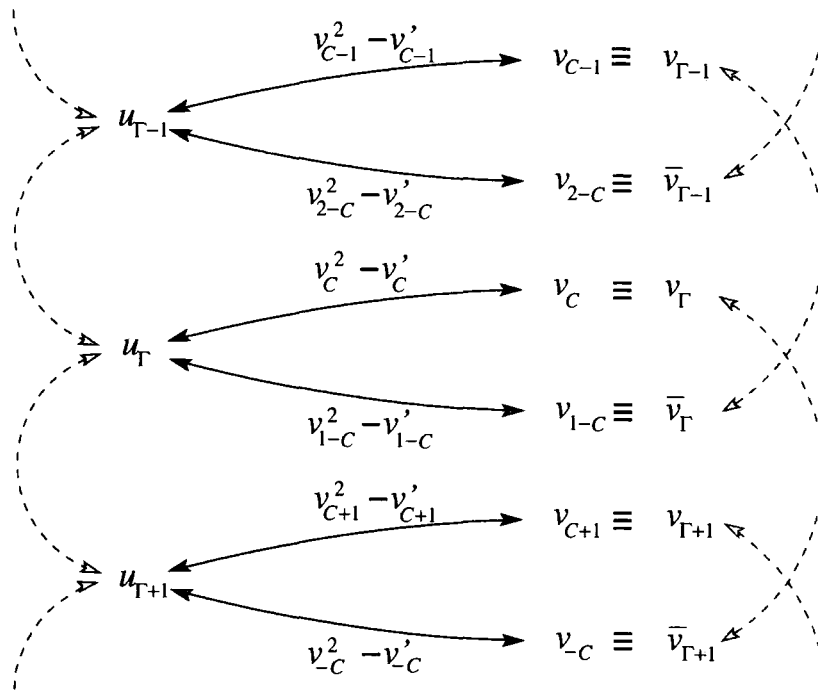


Figure 5.9: The structure of how the Miura map in combination with a sign flip on Painlevé II induces the Bäcklund transformations between u_{Γ} and $u_{\Gamma\pm 1}$.

$u_{\Gamma}(z)$ gives us information about backgrounds with Γ ZZ branes. So far we have not really studied the FZZT branes. In Section 3.8 we saw that, for the bosonic $(2, 2m-1)$ models, it was the wavefunction, ψ , itself that should be interpreted as the FZZT partition function, with boundary cosmological constant λ . Since then we have seen the Baker function come in useful in a multitude of contexts, but we have not yet attempted to analyse it in FZZT terms. It is natural to speculate that the Baker function also controls the FZZT branes in the Type 0A models. This was conjectured in [51], where λ was identified as the square of the boundary cosmological constant, μ_B , on the FZZT brane. Real values of μ_B therefore yield $\lambda \geq 0$, and we see that, since there are no bound states, the corresponding Baker functions must ‘blow-up’ at $z \rightarrow \infty$ or $z \rightarrow -\infty$ infinity, just like the solutions displayed in Figure 5.8 for $\lambda = 0$. Previously in the literature, the FZZT partition function as presented did not have an obvious interpretation in terms of string worldsheets. In the remainder of this chapter we will see a much more natural way of presenting matters that allows us to easily understand the physics involved and the effects of the boundary cosmological constant. At $\lambda = 0$ we have seen that the wavefunctions, $\psi(z)$, $\bar{\psi}(z)$, are intimately related to the ZZ branes in the background. We will therefore start our study of



the FZZT branes with the $\lambda = 0$ case in order to try to understand the physical interpretation of the Bäcklund transformation. In Section 5.3 we will move on to study the effects of non-zero λ .

5.2.1 The Wavefunction as a Probe FZZT Brane

Recall that $v(z) = -\psi'/\psi$. Since ψ has an interpretation as the FZZT partition function, then the natural interpretation of $v(z)$ is as the first derivative of the *connected* FZZT partition function, or FZZT free energy. We will see that this interpretation is correct. Let us start in the $z \rightarrow +\infty$ limit (recall $v_{C_+} \equiv v_\Gamma$, $v_{C_-} \equiv \bar{v}_\Gamma$), where from (5.26) we have:

$$v_{C_\pm} = \pm z^{\frac{1}{2}} + \frac{1}{2} \frac{(\frac{1}{2} \pm \Gamma)}{z} \mp \frac{1}{32} \frac{\nu^2}{z^{5/2}} (12\Gamma^2 \pm 12\Gamma + 5) + \dots \quad (5.36)$$

Integrating once and dividing by ν gives the FZZT free energy:

$$\begin{aligned} \mathcal{F} &= \frac{2}{3} \frac{z^{\frac{3}{2}}}{\nu} + \frac{1}{2} \left(\frac{1}{2} + \Gamma \right) \ln z + \frac{1}{48} \frac{\nu^2}{z^{3/2}} (12\Gamma^2 + 12\Gamma + 5) + \dots \\ &= \frac{2}{3} g_s^{-1} + \frac{1}{2} \left(\frac{1}{2} + \Gamma \right) g_s^0 \ln z + \frac{1}{48} g_s^1 (12\Gamma^2 + 12\Gamma + 5) + \dots, \\ \bar{\mathcal{F}} &= -\frac{2}{3} g_s^{-1} + \frac{1}{2} \left(\frac{1}{2} - \Gamma \right) g_s^0 \ln z - \frac{1}{48} g_s^1 (12\Gamma^2 - 12\Gamma + 5) + \dots \end{aligned} \quad (5.37)$$

By rewriting these expansions in terms of $g_s = \frac{\nu}{\mu^{3/2}}$, we can again make a worldsheet interpretation, but to do so we must include *two types* of boundary. Once again, worldsheets with boundaries ending on ZZ D-branes have a factor of Γ associated to them, but there is now another type of boundary associated with a single probe FZZT brane. This sum of connected diagrams (all with at least one boundary on the FZZT D-brane) is evidently the free energy of the FZZT D-brane in the presence of the Γ background ZZ D-branes. We can therefore write a term with b ZZ boundaries, f FZZT boundaries and h handles in the form $\Gamma^b g_s^{-\chi}$, with Euler number $\chi = 2 - 2h - b - f$. Exponentiation to form the wavefunction ψ is then the construction of the partition function of this system [41, 51]. It turns out that starting from the $v_\Gamma(z)$ expansion, one can trivially obtain the $\bar{v}(z)$ expansion by multiplying each term in the expansion by $(-1)^f$.

In Chapter 3 we explained how the ZZ branes are localised in the strong coupling regime at $\varphi \rightarrow \infty$, where φ is the Liouville direction. The FZZT D-branes on the other hand are extended in φ , but dissolve and come to an end at a specific value

$\varphi = \varphi_\lambda$, where $\varphi_\lambda \sim -\ln \lambda$. So the case $\lambda = 0$, which we have been studying so far, is the extreme case of extending the end of the FZZT D-brane probe all the way up to touch the Γ ZZ D-branes residing at $\varphi = +\infty$. This explains rather nicely the form of the expansion that we obtain from the Painlevé II equation in this situation. We have all possible diagrams which start on the FZZT branes; ones which can end on the background D-branes, and ones which do not. See Figure 5.10. In particular, the leading diagram is just the disc diagram measuring the tension of the probe brane as $\tau_{\text{fzzt}} = \frac{2}{3}g_s^{-1}$.

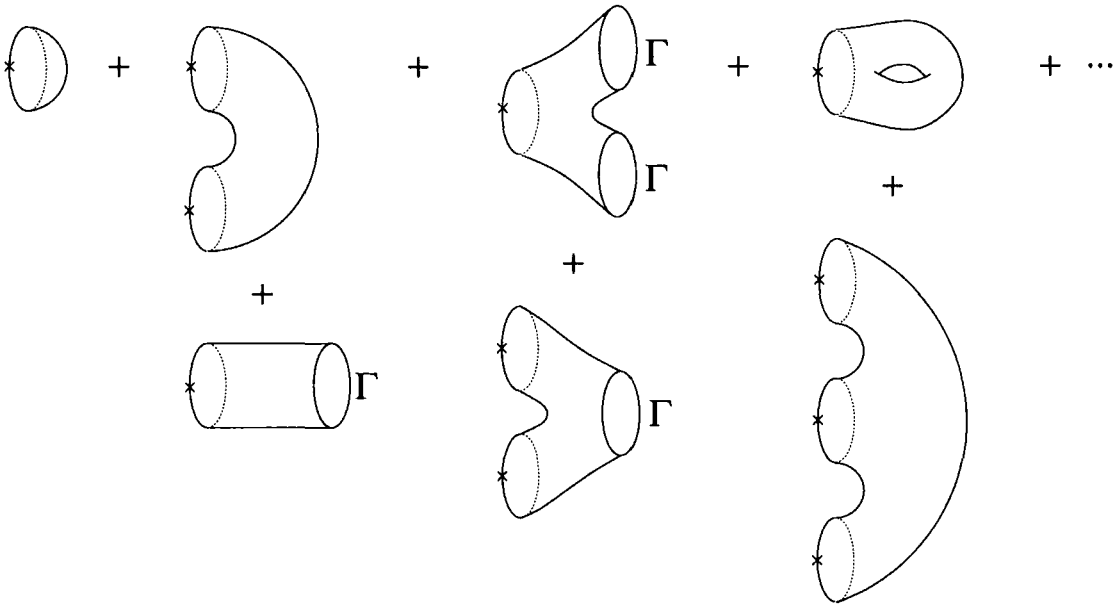


Figure 5.10: Some diagrams contributing from the worldsheet expansion (5.37) given by perturbatively solving the Painlevé II equation at large positive z . There is a single FZZT D-brane (boundaries ending on which are drawn with an ‘x’) probing a background of Γ ZZ D-branes (boundaries ending on which are drawn with a ‘ Γ ’).

There is also the limit $z \rightarrow -\infty$. Expanding v_{C_\pm} in this limit, dividing by ν and integrating once gives:

$$\mathcal{F} = -\left(\frac{1}{2} + \Gamma\right) g_s^0 \ln |z| + \frac{1}{24} g_s^2 (4\Gamma^2 - 1) (2\Gamma + 3) + \dots, \quad (5.38)$$

$$\bar{\mathcal{F}} = -\left(\frac{1}{2} - \Gamma\right) g_s^0 \ln |z| - \frac{1}{24} g_s^2 (4\Gamma^2 - 1) (2\Gamma - 3) + \dots. \quad (5.39)$$

Unlike the case of $u(z)$ itself, we are now finding odd powers of Γ appearing in this regime. This makes an interpretation in terms of a FZZT brane probing an R-R flux

background problematic. Therefore it looks as if we have no choice but to include ZZ brane boundaries in the worldsheets. However, in subsection 5.3.2 we will see a possible way to define a flux-only background.

5.2.2 Bäcklund Transformations For The FZZT Partition Function

Recall that the Bäcklund transformation for solutions, $u(z)$, to the string equation can be derived by combining the Miura map with the sign-flip symmetry of the Painlevé II equation (5.5), resulting in, for example:

$$u_{\Gamma+1} = v_{\Gamma}^2 + v'_{\Gamma}, \quad (5.40)$$

$$u_{\Gamma} = v_{\Gamma}^2 - v'_{\Gamma} = \bar{v}_{\Gamma}^2 - \bar{v}'_{\Gamma}, \quad (5.41)$$

$$u_{\Gamma-1} = \bar{v}_{\Gamma}^2 + \bar{v}'_{\Gamma}. \quad (5.42)$$

From the discussion under equation (5.30), and as is clear on the diagram in Figure 5.9, we also have the relations:

$$v_{\Gamma} = -\bar{v}_{\Gamma+1}, \quad (5.43)$$

$$\text{and } \bar{v}_{\Gamma} = -v_{\Gamma-1}, \quad (5.44)$$

which shall be useful later. Subtracting equation (5.40) from (5.41) yields:

$$u_{\Gamma} + 2v'_{\Gamma} = u_{\Gamma+1}, \quad (5.45)$$

while the difference of equations (5.42) and (5.41) yields:

$$u_{\Gamma} + 2\bar{v}'_{\Gamma} = u_{\Gamma-1}. \quad (5.46)$$

So it is as if adding two ($\lambda = 0$) FZZT branes given by v is equivalent to adding one ZZ brane; whilst adding two ($\lambda = 0$) FZZT branes given by \bar{v} has the effect of removing one ZZ D-brane. This is not strictly correct however, because it neglects the interaction between the two FZZT branes¹³. Another way of looking at things is to use equation (5.43) and (5.44) to rewrite equations (5.45), and (5.46) as:

$$u_{\Gamma+1} + \bar{v}'_{\Gamma+1} = u_{\Gamma} + v'_{\Gamma}, \quad (5.47)$$

¹³See [41] for a description of this in the bosonic case.

so the background with $(\Gamma + 1)$ ZZ branes (or flux units) and one \bar{v} -FZZT brane is the same as the background with Γ ZZ branes (or flux units) and one v -FZZT brane.

Recall from the discussion of supersymmetric quantum mechanics that the Miura map arises from factorising \mathcal{H} in the form:

$$\mathcal{H}_\Gamma \psi \equiv (-d^2 + u_\Gamma) \psi = (-d + v)(d + v) \psi. \quad (5.48)$$

A solution of this equation is of course just $\psi = e^p$, where $p' = -v$, since function p clearly satisfies $(d + v)e^p = 0$. However, there should also exist another function q also satisfying $\mathcal{H}_\Gamma e^q = 0$ but not $(d + v)e^q = 0$. Writing $e^r = (d + v)e^q$ we see that r satisfies the following two equations:

$$(-d + v)e^r = 0, \quad (5.49)$$

$$\mathcal{H}_{\Gamma+1} e^r \equiv (d + v)(-d + v)e^r = (-d^2 + u_{\Gamma+1})e^r = 0. \quad (5.50)$$

Using (5.49) we see that $r' = v$ is a solution. Therefore we can write:

$$e^r = (d + v)e^q = (d - r')e^q = (q' - r')e^q, \quad (5.51)$$

which implies the following auto-Bäcklund transformation between q and r :

$$e^{r+q} = q' - r', \quad (5.52)$$

where q' and r' are both solutions of the Painlevé II equation differing by unit Γ . This directly relates solution v_Γ to solutions $v_{\Gamma\pm 1}$; and similarly for \bar{v} . The Bäcklund transformation (5.52) is quite cumbersome to use in practice, but one can in fact derive a more explicit version [56] using the same equations that led us to the explicit version of the u transformation earlier (see Section D.1). We find:

$$\begin{aligned} v_{\Gamma-1} &= -v_\Gamma + \frac{2\nu\Gamma}{\mathcal{R}[v_\Gamma^2 - v'_\Gamma]}, & \bar{v}_{\Gamma-1} &= -\bar{v}_\Gamma - \frac{2\nu(\Gamma-1)}{\mathcal{R}[\bar{v}_\Gamma^2 + \bar{v}'_\Gamma]}, \\ v_{\Gamma+1} &= -v_\Gamma + \frac{2\nu(\Gamma+1)}{\mathcal{R}[v_\Gamma^2 + v'_\Gamma]}, & \bar{v}_{\Gamma+1} &= -\bar{v}_\Gamma - \frac{2\nu\Gamma}{\mathcal{R}[\bar{v}_\Gamma^2 - \bar{v}'_\Gamma]}, \end{aligned} \quad (5.53)$$

Using the fact that $v_0 = \bar{v}_0$ it is easy to show inductively that $v_\Gamma = \bar{v}_{-\Gamma}$ if and only if Γ is an integer. This is borne out numerically in the same way¹⁴ that we showed $u_\Gamma = u_{-\Gamma}$ for integer Γ (see Figures 5.6, and 5.7). We now observe that this fits well

¹⁴That is, starting with $\Gamma = \varepsilon$, Bäcklund transforming to $\Gamma = -1 + \varepsilon$, and letting $\varepsilon \rightarrow 0$.

with the earlier observation that the wave equation, (5.2), in the $z \rightarrow -\infty$ limit is just (after a simple change of variables) Bessel's equation with $n = \Gamma$.

These results are of course consistent:

$$u_\Gamma = v_\Gamma^2 - v'_\Gamma = \bar{v}_\Gamma^2 - \bar{v}'_\Gamma = v_{-\Gamma}^2 - v'_{-\Gamma} = u_{-\Gamma}. \quad (5.54)$$

So, in many ways it is as if v is an FZZT brane and \bar{v} is an anti-FZZT brane, with charge half that of a ZZ brane. The only way that this is consistent under charge conjugation, $\Gamma \mapsto -\Gamma$, (under which the physics should be invariant) is for integer Γ . This is yet more evidence that Γ is quantised. This differs from the interpretation made in [51], which was done mainly from a Type 0B perspective. There the authors did not identify the function $\bar{\psi}(z)$; so instead the function $\psi(z)$ was identified as a neutral brane.

The most intuitive interpretation of the physics of $v(z)$ and $\bar{v}(z)$ is to consider turning on λ (we will do this explicitly below). Then, the results of (5.47) do not hold because, as we have seen all along, it is only for $\lambda = 0$ that the FZZT branes and ZZ branes are intimately related. Recall that a FZZT brane with non-zero λ stretches along the Liouville direction from $\varphi = -\infty$ and dissolves at $\varphi_\lambda \sim -\ln \lambda$. As we decrease λ , the tip of the probe FZZT brane stretches out towards $\varphi = \infty$ where Γ ZZ branes reside. When λ is finally reduced to zero the FZZT brane tip can make contact with the ZZ branes and (5.47) becomes valid. Then, instead of an FZZT brane with Γ ZZ branes, we can instead think of the system as an anti-FZZT brane with $(\Gamma + 1)$ ZZ branes. We see that as the original FZZT brane reaches $\varphi = \infty$ it has deposited an extra ZZ brane and been converted to an anti-FZZT brane. Turning on λ again, its tip retreats along the Liouville direction. This interpretation was also made for the Type 0B charged FZZT branes in [51]. Notice also that this ties in with the discussion in Section 3.4, where FZZT branes and ZZ branes were related. A perusal of that section will confirm that the ZZ branes associated with Γ must be labelled by $(r, s) = (p/2, q/2)$, since $\lambda = 0$ (and hence $\mu_B = 0$). It would be interesting to uncover ZZ branes with other labels, (r, s) , and hence other values of μ_B , in the integrable system context, but this has not been done as yet. It is possible that they have values of μ_B that make them unphysical.

An interesting observation is that if one adds v_Γ and \bar{v}_Γ together asymptotically

then one obtains:

$$\begin{aligned}
 v_\Gamma + \bar{v}_\Gamma &= \frac{\nu}{2z} - \frac{3\nu^2\Gamma}{4z^{5/2}} + \frac{15\nu^3}{32z^4} + \frac{3\nu^4\Gamma^2}{2z^4} - \frac{\nu^5\Gamma(420\Gamma^2 + 459)}{z^{11/2}} + \dots \quad z \rightarrow \infty \\
 v_\Gamma + \bar{v}_\Gamma &= -\frac{\nu}{z} - \frac{\nu^3(24\Gamma^2 - 3)}{4z^4} - \frac{\nu^5(240\Gamma^4 - 504\Gamma^2 + 111)}{16z^7} + \dots \quad z \rightarrow -\infty
 \end{aligned} \tag{5.55}$$

This can be interpreted entirely in terms of a background containing just ZZ branes with no FZZT branes. So it is almost as if a brane and an anti-brane have annihilated in some way. We have again ignored any interaction term between the two branes here, but it is interesting and worthy of note nonetheless.

5.3 FZZT Branes With Non-Zero Boundary Cosmological Constant

Let us now consider the case of $\lambda \neq 0$, and examine the structure of $\psi(z)$ at finite energy. For this it is harder to get an exact non-linear ordinary differential equation (another deformation of Painlevé II for pure supergravity) but still possible. A better way to calculate it is to do as we did for $\lambda = 0$ and calculate v_Γ and \bar{v}_Γ directly from the Miura map, only this time modified to include λ :

$$u = v^2 - v' - \lambda, \tag{5.56}$$

5.3.1 Large Positive z

Returning to the case $m = 1$, we get for example (as $z \rightarrow +\infty$):

$$\begin{aligned}
 v_\Gamma(z) &= z^{1/2} + \frac{1}{2} \frac{\lambda}{z^{1/2}} + \left(\frac{1}{4} + \frac{\Gamma}{2} \right) \frac{\nu}{z} - \frac{1}{8} \frac{\lambda^2}{z^{3/2}} - \frac{\lambda}{4} (\Gamma + 1) \frac{1}{z^2} \\
 &\quad + \frac{1}{32} (2\lambda^3 - 5 - 12\Gamma^2 - 12\Gamma) \frac{1}{z^{5/2}} + \frac{\lambda^2}{16} (4 + 3\Gamma) \frac{1}{z^3} \\
 &\quad + \frac{5}{128} \lambda (8\Gamma^2 + 16\Gamma + 10 - \lambda^3) \frac{1}{z^{7/2}} \\
 &\quad + \left(\frac{1}{2}\Gamma^3 + \frac{3}{4}\Gamma^2 + \frac{23}{32}\Gamma + \frac{15}{64} - \frac{1}{4}\lambda^3 - \frac{5}{32}\lambda^3\Gamma \right) \frac{\nu^2}{z^4} + \dots \tag{5.57}
 \end{aligned}$$

On the face of it, there is some difficulty in interpreting these terms in the familiar language of string perturbation theory. However, recall that λ is proportional to the square of the boundary cosmological constant, μ_B , on the FZZT brane. This

weights boundaries according to their length. We therefore see that the effect of λ to the contribution of a particular worldsheet should be dependent upon the number of boundaries, f , that the worldsheet has ending on the FZZT brane. We know that μ_B has units of inverse length and μ has units of inverse area. Accordingly, we see that λ and z should have the same units. Therefore, the natural guess is that λ will renormalise z additively in some way that depends upon f . Let us re-examine the expansion of (5.57) with this in mind. Restricting to terms without powers of ν or Γ (which we expect to correspond to the renormalised disc) we find that we can write:

$$z^{1/2} \left(1 + \frac{1}{2} \frac{\lambda}{z} - \frac{1}{8} \left(\frac{\lambda}{z} \right)^2 + \frac{1}{16} \left(\frac{\lambda}{z} \right)^3 + \dots \right) = z^{1/2} \left(1 + \frac{\lambda}{z} \right)^{1/2} = (z + \lambda)^{1/2}. \quad (5.58)$$

Integrating once and dividing by ν to find the free energy gives:

$$F_{(0,1,0)} = \frac{2}{3} \tilde{g}_s^{-1}, \quad \text{where} \quad \tilde{g}_s = \frac{\nu}{(z + \lambda)^{\frac{3}{2}}}, \quad (5.59)$$

where we have introduced the notation $F_{(b,f,h)}$ to denote the contribution to the free energy from a diagram with f FZZT boundaries, b ZZ boundaries and h handles. This resummation turns out to be precisely what happens to all diagrams involving pure FZZT boundaries with no handles or ZZ boundaries. For example, let us look at the case of 3 FZZT boundaries. After some algebra, the infinite series of λ contributions can be resummed, with the result:

$$F_{(0,3,0)} \sim \frac{\nu}{(z + \lambda)^{\frac{3}{2}}} \sim \tilde{g}_s. \quad (5.60)$$

The diagram with 1 FZZT boundary and one handle also gives a result of the same form, with the total contribution being¹⁵:

$$F_{(0,3,0)} + F_{(0,1,1)} = \frac{5}{48} \tilde{g}_s. \quad (5.61)$$

Turning to diagrams with both FZZT and ZZ boundaries, let us see what happens to the three-string vertices. We get, for the case of 2 FZZT boundaries and 1 ZZ, the result:

$$F_{(1,2,0)} = \frac{\nu \Gamma}{4(z + \lambda)z^{1/2}}, \quad (5.62)$$

¹⁵Because we can only distinguish terms by their factors of g_s and Γ , it seems that there is no obvious way to separate $F_{(b,f,h)}$ and $F_{(b,f-2,h+1)}$ in the expansion.

whilst the case of 1 FZZT and 2 ZZ yields:

$$F_{(2,1,0)} = \frac{\nu\Gamma^2}{4(z+\lambda)^{1/2}z}. \quad (5.63)$$

For these two terms the pattern appears to be $z^{-b/2}(z+\lambda)^{-f/2}$. That is, for every FZZT boundary one indeed finds that z is renormalised by λ . This renormalisation is consistent with the fact that λ sets the separation from the FZZT D-brane's tip to the Γ ZZ D-branes in the background.

Let us consider the structure at higher order in perturbation theory. The contribution from four FZZT boundaries mixes with that of the diagram with two FZZT boundaries and one handle to give, after resumming:

$$F_{(0,4,0)} + F_{(0,2,1)} = -\frac{5\nu^2}{64} \left(\frac{1}{(z+\lambda)^3} \right) = -\frac{5}{64} \tilde{g}_s^2. \quad (5.64)$$

More interesting are the cases mixing the different types of boundaries. For $F_{(2,2,0)}$ we expect (since $b=f$) that half the factors contributing would be z and half would be $(z+\lambda)$. Since this order leads with a z^{-4} term in $v(z)$, we could hazard a guess at $z^{-2}(z+\lambda)^{-2}$. However, this is not correct. Instead we find that there are two terms contributing:

$$F_{(2,2,0)} = -\frac{\nu^2\Gamma^2}{8} \left[\frac{1}{(z+\lambda)^2z} + \frac{1}{(z+\lambda)z^2} \right]. \quad (5.65)$$

Inspection of this result confirms that it is actually proportional to $[z(z+\lambda)]'$. This is certainly more elegant than it being a sum of two terms, and it is natural to conjecture (bearing mind all the resummations we have discussed so far) that the other terms with no handles can be written as proportional to $[z^{-b/2}(z+\lambda)^{-f/2}]^{(b+f-3)}$, where the superscript in parentheses denotes $b+f-3$ differentiations with respect to z . An analysis of the relevant terms in the expansion confirms that this conjecture appears to be correct.

Matters become more complicated when when considers terms with handles as well as boundaries. So far we have only made partial progress in this area, which we will describe below. To do this it will be helpful to introduce the following shorthand notation:

$$(b, f) \equiv z^{-b/2}(z+\lambda)^{-f/2}, \quad (5.66)$$

It is clear that we have $(b, f)' = -\frac{b}{2}(b+2, f) - \frac{f}{2}(b, f+2)$. In the remainder of this section we will define the contribution to the free energy from a particular diagram,

$F_{(b,f,h)}$, implicitly to have a factor of $\nu^{-\chi}\Gamma^b$ multiplying it, where $\chi \equiv 2 - 2h - b - f$ is the Euler number. Once again we will consider the $v_\Gamma(z)$ expansion; although the $\bar{v}_\Gamma(z)$ expansion can be trivially obtained by multiplying the $F_{(b,f,h)}$ by $(-1)^f$.

For a surface with no handles ($h=0$), we find that its contribution to the free energy in expansion can be resummed (up to an overall constant) into the form:

$$F_{(b,f,0)} \propto (b, f)^{(b+f-3)} \equiv \langle b, f \rangle. \quad (5.67)$$

If the number of derivatives is negative in (5.67), as is the case with the disc and the cylinder, then one should integrate instead of differentiating. A similar equation to (5.67) holds in the case of h non-zero providing that one or other of b and f is zero instead. We have:

$$\begin{aligned} F_{(b,0,h)} &\propto (b, 0)^{(b+h-1)} \equiv (b + 2h, 0)^{(b-1)}, \quad b, h \neq 0, \\ F_{(0,f,h)} &\propto (0, f)^{(f+h-1)} \equiv (0, f + 2h)^{(f-1)}, \quad f, h \neq 0. \\ F_{(0,0,h)} &\propto (6h - 4, 0)^{(-1)}. \end{aligned} \quad (5.68)$$

When all of b , f and h are non-zero simultaneously then matters become more complicated. It turns out to be possible to resum every contribution to the free energy as a sum of terms of the form (b, f) . The number of terms increases with the ‘complexity’ of the series, and the type of terms present in each resummation follow a strict and predictable pattern. Below (in equations (5.69), (5.70) and (5.71)) is a list of the first several resummed terms of the $v(z)$ expansion including overall coefficients and written in the standard form of (5.67) and (5.68) wherever possible. That these terms are a mixture of surfaces can often be seen by the relative lack of simplicity of their coefficients.

$$\begin{aligned} F''_{(0,1,0)} &= \frac{1}{2}(0, 1), \quad F'_{(0,2,0)} = \frac{1}{4}(0, 2), \quad F'_{(1,1,0)} = \frac{1}{2}(1, 1) \\ F_{(0,3,0)} + F_{(0,1,1)} &= \frac{5}{48}(3, 0), \quad F_{(1,2,0)} = \frac{1}{4}(1, 2), \quad F_{(2,1,0)} = \frac{1}{4}(2, 1), \\ F_{(0,4,0)} + F_{(0,2,1)} &= \frac{5}{128}(0, 4)', \quad F_{(2,2,0)} = \frac{1}{8}(2, 2)', \quad F_{(3,1,0)} = \frac{1}{12}(3, 1)' \\ F_{(1,3,0)} + F_{(1,1,1)} &= -\frac{5}{32}(1, 5) - \frac{5}{96}(3, 3) - \frac{1}{32}(5, 1), \end{aligned} \quad (5.69)$$

$$\begin{aligned}
 F_{(0,5,0)} + F_{(0,3,1)} + F_{(0,1,2)} &= \frac{221}{16128}(0, 5)'' \\
 F_{(1,4,0)} + F_{(1,2,1)} &= \frac{15}{64}(1, 8) + \frac{5}{64}(3, 6) + \frac{3}{64}(5, 4) + \frac{5}{128}(7, 2), \\
 F_{(2,3,0)} + F_{(2,1,1)} &= \frac{25}{128}(2, 7) + \frac{5}{32}(4, 5) + \frac{23}{192}(6, 3) + \frac{7}{64}(8, 1), \\
 F_{(3,2,0)} &= \frac{1}{24}(3, 2)'', \quad F_{(4,1,0)} = \frac{1}{48}(4, 1)'', \\
 F_{(0,6,0)} + F_{(0,4,1)} + F_{(0,2,2)} &= \frac{113}{24576}(0, 6)^{(3)} \tag{5.70}
 \end{aligned}$$

$$\begin{aligned}
 F_{(1,5,0)} + F_{(1,3,1)} + F_{(1,1,2)} &= -\frac{1105}{2048}(1, 11) - \frac{1105}{6144}(3, 9) - \frac{221}{2048}(5, 7) \\
 &\quad - \frac{165}{2048}(7, 5) - \frac{385}{6144}(9, 3) - \frac{105}{2048}(11, 1), \\
 F_{(2,4,0)} + F_{(2,2,1)} &= -\frac{15}{32}(2, 10) - \frac{45}{128}(4, 8) - \frac{69}{256}(6, 6) \\
 &\quad - \frac{29}{128}(8, 4) - \frac{7}{32}(10, 2), \\
 F_{(3,3,0)} + F_{(3,1,1)} &= -\frac{175}{768}(3, 9) - \frac{75}{256}(5, 7) - \frac{39}{128}(7, 5) \\
 &\quad - \frac{223}{768}(9, 3) - \frac{83}{256}(11, 1). \tag{5.71}
 \end{aligned}$$

It is obvious from equations (5.69), (5.70) and (5.71) that it is easy to predict the terms that will be present in higher order worldsheet contributions. The coefficients can then be determined by fitting to the form of the expansion. However, we feel that it should also be possible to determine the relative coefficients within any given contribution from some simple underlying set of rules, since the resummations in (5.69)–(5.71) are rather messy. By this we mean that a surface's specific contribution to the free energy should be able to be isolated, up to an overall constant. A natural hypothesis arises by considering the diagrams associated with $\langle b, f \rangle$ as in some way *fundamental* (see Figure 5.11); then trying to build handled diagrams by *stitching* these fundamental diagrams together by joining boundaries. The obvious guess for how to represent the joining of two surfaces, $\langle b_1, f_1 \rangle$ and $\langle b_2, f_2 \rangle$, together is multiplication: $\langle b_1, f_1 \rangle \langle b_2, f_2 \rangle$.

It is clear from the form of equations (5.69)–(5.71) that, for this to work, one must only stitch like boundaries together: that is ZZ boundaries to ZZ boundaries and FZZT boundaries to FZZT boundaries. Notice that if f is zero then surfaces with handles have no λ dependence; whereas if b is zero then surfaces with handles only depend on the combination $(z + \lambda)$. If these handled terms are formed from the stitching of fundamental surfaces, then those with $f = 0$ must be formed exclusively

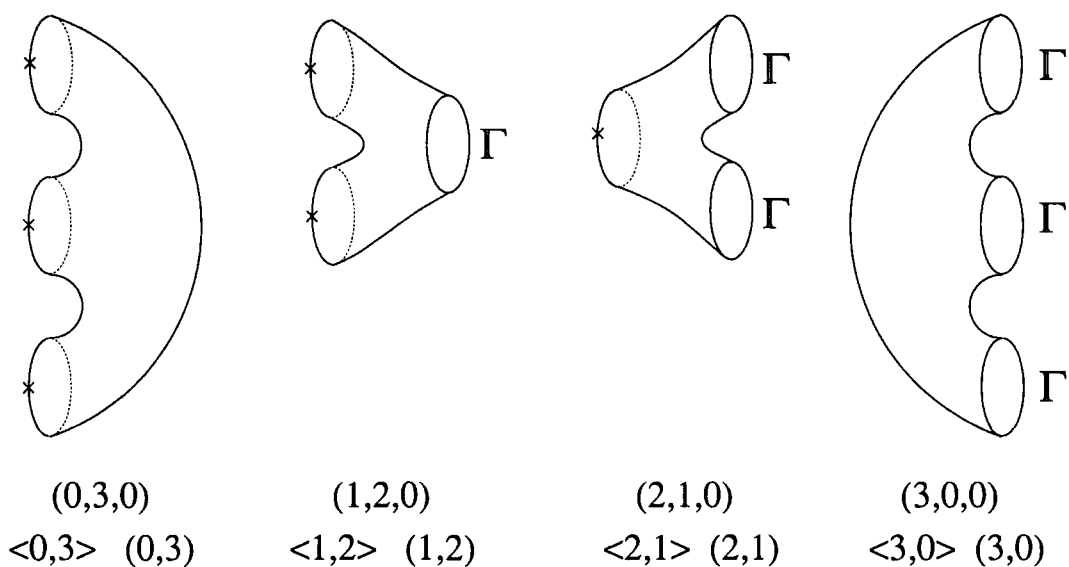


Figure 5.11: Some of the diagrams fundamental to building diagrams of higher topology (and their contributions the free energy) using the methods described in the text. Shown is some of the notation used to describe them in the text in general $((b, f, h))$, to denote their fundamental status $((b, f))$ and also the shorthand for the terms which appear in the free energy $((b, f))$, see equation (5.66)).

from ZZ–ZZ stitchings; those with $b = 0$ exclusively from FZZT–FZZT stitchings. Let us explore this by attempting to construct surfaces with $f = 1$. Since b is arbitrary, these surfaces are close in structure to the $f = 0$ case, and so we will try to construct them (at least initially) from ZZ–ZZ stitchings only. Consider the combined $(1, 3, 0)$ and $(1, 1, 1)$ term. Equation (5.67) tells us the expected form of the former term, and we can denote it as $\langle 1, 3 \rangle$. Inspection of Figure 5.12 tells us that the $(1, 1, 1)$ term can only be constructed from the stitching of two pairs of ZZ boundaries between the fundamental three-string surfaces $\langle 3, 0 \rangle$ and $\langle 2, 1 \rangle$.

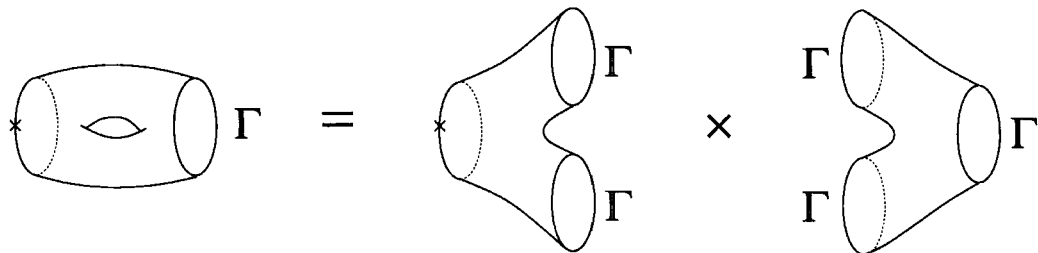


Figure 5.12: Schematic representation of the stitching of diagrams associated to multiplication to recover terms in the free energy. (See text for details.)

Our guess is therefore:

$$F_{(1,3,0)} + F_{(1,1,1)} = a_0 \langle 1, 3 \rangle + b_{00} \langle 2, 1 \rangle \langle 3, 0 \rangle = a_0 (1, 3)' + b_{00} (2, 1) (3, 0). \quad (5.72)$$

Comparison with equations (5.69) yields the result $a_0 = 5/48$, $b_{00} = -1/32$. Moving on to the combined $(1, 4, 0)$ and $(1, 2, 1)$ term we predict:

$$\begin{aligned} F_{(2,3,0)} + F_{(2,1,1)} &= a_1 \langle 2, 3 \rangle + b_{10} \langle 2, 1 \rangle \langle 4, 0 \rangle + b_{11} \langle 3, 0 \rangle \langle 3, 1 \rangle \\ &= a_2 (2, 3)'' + b_{10} (2, 1) (4, 0)' + b_{11} (3, 0) (3, 1)'. \end{aligned} \quad (5.73)$$

The justification for this ansatz is that it contains every pair of constituent surfaces that could possibly contribute. Again, our prediction can be realised with $a_1 = 5/96$, $b_{10} = b_{11} = -1/32$. At the next level we have:

$$\begin{aligned} F_{(3,3,0)} + F_{(3,1,1)} &= a_2 \langle 3, 3 \rangle + b_{20} \langle 2, 1 \rangle \langle 5, 0 \rangle + b_{21} \langle 3, 1 \rangle \langle 4, 0 \rangle + b_{22} \langle 3, 0 \rangle \langle 4, 1 \rangle \\ &= a_2 (3, 3)''' + b_{20} (2, 1) (5, 0)'' + b_{21} (3, 1)' (4, 0)' + b_{22} (3, 0) (4, 1)''. \end{aligned} \quad (5.74)$$

This time we find that $a_2 = 5/288$, $b_{21} = 2b_{20} = 2b_{22} = -1/32$. We can now see a possible pattern emerging amongst the b_{ij} :

$$b_{ij} = b_i \binom{i}{j} \quad b_0 = -\frac{1}{32}, \quad b_1 = -\frac{1}{32}, \quad b_2 = -\frac{1}{64}. \quad (5.75)$$

Testing this conjecture for the combined $(4, 3, 0)$ and $(4, 1, 1)$ term we find that it does indeed hold, with $a_3 = 5/1152$, $b_3 = -1/192$. So it seems that we have uncovered a general rule:

$$F_{(k,3,0)} + F_{(k,1,1)} = a_k \langle k, 3 \rangle + b_k \sum_{i=0}^k \binom{k}{i} \langle 3+i, 0 \rangle \langle 2+k-i, 1 \rangle. \quad (5.76)$$

The origin of the binomial factors here is not known. They may relate to the number of ways that the constituent worldsheets can be stitched together, but we have not been able to understand how this would work. Notice that upon including the z derivatives in (5.76), by replacing the \langle, \rangle with $(,)$ everywhere, the resultant equation is reminiscent of the Liebniz rule for the multiple differentiation of a product. Note also that it seems that we can now associate the a_i as being the coefficients of the $(i+1, 3, 0)$ and the b_i as being the coefficients of the $(i+1, 1, 1)$. Matters are not this simple however, as will be shown when we analyse other surfaces below.

We return now to the combined $(1, 4, 0)$ and $(1, 2, 1)$ term. Since the handled term here fits into the $b=1$ subclass, analogous to the $f=1$ subclass explored above, we make the natural guess that it can be constructed from exclusively FZZT-FZZT stitchings. It turns out that this works providing that we add in an extra term proportional to $(3, 2, 1)$. This can be thought of as self-stitching of two ZZ boundaries on the worldsheet to form a handle. We find:

$$\begin{aligned} F_{(1,4,0)} + F_{(1,2,1)} &= A_1 \langle 1, 4 \rangle + B_{10} \langle 1, 2 \rangle \langle 0, 4 \rangle + B_{11} \langle 0, 3 \rangle \langle 1, 3 \rangle + C_1 \langle 3, 2 \rangle \\ &= A_1 (1, 4)'' + B_{10} (1, 2) (0, 4)' + B_{11} (0, 3) (1, 3)' + C_1 (3, 2)'', \end{aligned} \quad (5.77)$$

where $A_1 = 1/48$, $B_{10} = B_{11} = -1/32$, $C_1 = 1/96$. This seems to be the only sensible choice: all other schemes we tried yielded coefficients as unilluminating as those in the original resummation equations (5.69), (5.70) and (5.71). We can make some sense of this result if we consider the action of the transformation $\langle b, f \rangle \leftrightarrow \langle f, b \rangle$ on the resummed contributions. We will denote the action of this transformation on a surface (b, f, h) as $(b, f, h)^\dagger$. Note that this transformation is defined at the level of the $\langle b, f \rangle$ themselves, so the relation $(b, f, h)^\dagger \equiv (f, b, h)$ is not trivially satisfied. Indeed, we can immediately see a counter example using (5.68): $F_{(0,0,h)} \propto (6h-6, 0) \neq F_{(0,0,h)^\dagger} \propto (0, 6h-6)$. These $(0, 0, h)$ terms are from the background $u(z)$ expansion of course; but perhaps terms from the $v(z)$ expansion do satisfy $(b, f, h)^\dagger \equiv (f, b, h)$? From (5.67) we clearly see that surfaces without handles certainly do. By requiring that $(1, 2, 1)^\dagger = (2, 1, 1)$ we can unambiguously separate the handled terms from the non-handled terms (using (5.77) and (5.73)):

$$\begin{aligned} F_{(2,1,1)} &= \frac{1}{96} \langle 2, 3 \rangle - \frac{1}{32} \langle 2, 1 \rangle \langle 4, 0 \rangle - \frac{1}{32} \langle 3, 0 \rangle \langle 3, 1 \rangle, \\ F_{(1,2,1)} = F_{(2,1,1)^\dagger} &= \frac{1}{96} \langle 3, 2 \rangle - \frac{1}{32} \langle 1, 2 \rangle \langle 0, 4 \rangle - \frac{1}{32} \langle 0, 3 \rangle \langle 1, 3 \rangle, \end{aligned} \quad (5.78)$$

and, as such:

$$F_{(2,3,0)} = \frac{1}{24} \langle 2, 3 \rangle, \quad F_{(1,4,0)} = \frac{1}{48} \langle 1, 4 \rangle. \quad (5.79)$$

This would explain the appearance of the self-stitched $(3, 2, 1)$ term in (5.77). Using the same procedure we can render the components of (5.72) in manifestly invariant form with $(1, 1, 1)^\dagger = (1, 1, 1)$:

$$\begin{aligned} F_{(1,1,1)} &= \frac{1}{48} \langle 1, 3 \rangle + \frac{1}{96} \langle 1, 2 \rangle \langle 2, 1 \rangle + \frac{1}{48} \langle 3, 1 \rangle, \\ F_{(1,3,0)} &= \frac{1}{12} \langle 1, 3 \rangle. \end{aligned} \quad (5.80)$$

The $(1, 1, 1)$ term here now represents two self-stitched surfaces; plus one of each of FZZT-FZZT and ZZ-ZZ stitchings between the fundamental surfaces $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$.

Whether other surfaces are also invariant under the $b \leftrightarrow f$ operation remains to be seen. At higher orders things get much more difficult because of mixing of many different terms with various numbers of handles. The pattern cannot be binomial in every case because for many surfaces there are more possible fundamental stitchings that can contribute to the free energy than there are binomial coefficients. So far we have not been able to complete the elegant picture presented above for arbitrary (b, f, h) , though it is likely that the final solution will be extremely simple and symmetrical. This is work in progress. However, if all the surfaces making up $v(z)$ are indeed invariant under $b \leftrightarrow f$, then we can automatically generate the $(1, f, 1)$ series from the $(b, 1, 1)$ series (5.76). Regrettably, the next term in the $(1, f, 1)$ series, $(1, 3, 1)$, also mixes with a two-handle term, $(1, 1, 2)$, and so without knowing the appropriate two-handle rules we have been unable to test our conjecture.

5.3.2 Large Negative z

Let us now turn to the regime $z \rightarrow -\infty$ at non-zero λ . The expansion we obtain is:

$$\begin{aligned} v(z) = & \lambda^{1/2} + \frac{\nu^2(4\Gamma^2 - 1)}{8\lambda^{1/2}z^2} - \frac{\nu^3(4\Gamma^2 - 1)}{8\lambda z^3} - \frac{\nu^4(16\Gamma^4 - 104\Gamma^2 + 25)}{128\lambda^{3/2}z^4} \\ & + \frac{\nu^4(32\Gamma^4 - 80\Gamma^2 + 18)}{32\lambda^{1/2}z^5} + \frac{\nu^5(16\Gamma^4 - 56\Gamma^2 + 13)}{32\lambda^2z^5} \\ & - \frac{\nu^5(2560\Gamma^4 - 6400\Gamma^2 + 1440)}{1024\lambda z^6} + \frac{\nu^6(64\Gamma^6 - 1840\Gamma^4 + 4768\Gamma^2 - 1073)}{1024\lambda^{5/2}z^6} + \dots, \end{aligned} \quad (5.81)$$

where the sign of the square-root of λ determines whether this solution corresponds to v_Γ or \bar{v}_Γ . The first point of note is the that this expansion does not reduce to the expansion (5.26) that we had at $\lambda = 0$. In fact, at $\lambda = 0$ it is singular. We shall try to understand the physics of this. Integrating once and dividing by ν to obtain the free energy we find that the first few terms are:

$$F = \lambda^{\frac{1}{2}} \frac{z}{\nu} + \frac{(4\Gamma^2 - 1)}{8} \left(-\frac{\nu}{\lambda^{\frac{1}{2}}z} + \frac{\nu^2}{2\lambda z^2} \right) + \dots, \quad (5.82)$$

which fits with the asymptotic form of the wavefunction following from modified Bessel functions, observed in (5.25). We see that for the free energy to be real we must have $\lambda > 0$. This means that the $\lambda < 0$ oscillatory-wave solutions are not relevant in the string theory. This is as expected because $\lambda < 0$ corresponds to imaginary values of

the boundary cosmological constant. We have seen that there are no bound states for integer Γ ; so the corresponding wavefunctions must again ‘blow-up’ at $z \rightarrow \infty$ or $z \rightarrow -\infty$.

Notice that nowhere in (5.82) do we see a natural occurrence of the dimensionless coupling $g_s = \nu/z^{3/2}$, nor do we see the renormalised version $\tilde{g}_s = \nu/(z + \lambda)^{3/2}$. This seems to be a puzzle. The resolution is simply that we need not expect either g_s or \tilde{g}_s to appear as the natural stringy expansion parameter in this situation. Part of the reason is that λ and z appear very differently in this regime as compared to the large positive z regime. There, even when λ was zero, at tree level there was a natural parameter which gave Boltzmann weight to boundary loops, and this was z , the tree level part of $u(z)$ [60–62]. Introducing λ brings it in to perform a role already performed by z (at tree level) and so it simply shifts z as we have seen. In this large negative z regime, we have completely different behaviour. The potential $u(z)$ *vanishes* at leading order and so at that order there is nothing weighting the length of loops when $\lambda = 0$. So when λ is non-zero, there is now a weighting parameter at tree level, and so the natural loop expansion that it controls need not have anything to do with that of the $\lambda = 0$ case.

Correspondingly, we shall therefore not expect that the natural expansion parameter is g_s or \tilde{g}_s . Instead, an examination of the expression for the free energy shows that the natural expansion parameter is:

$$\hat{g}_s = \frac{\nu}{\lambda^{\frac{1}{2}} z}. \quad (5.83)$$

Notice that this is also a dimensionless combination of the important parameters in the problem, a combination which is not available when λ vanishes. Using this result we can write the free energy as:

$$\begin{aligned} F = & \hat{g}_s^{-1} - \left(\Gamma^2 - \frac{1}{4} \right) \left[\frac{1}{2} \hat{g}_s^1 + \frac{1}{4} \hat{g}_s^2 + \hat{g}_s^3 \left(-\frac{4\Gamma^2 - 25}{96} + \frac{\lambda}{z} \frac{4\Gamma^2 - 9}{16} \right) \right. \\ & + \hat{g}_s^4 \left(-\frac{4\Gamma^2 - 13}{32} + \frac{\lambda}{z} \frac{4\Gamma^2 - 9}{8} \right) \\ & + \hat{g}_s^5 \left(-\frac{(16\Gamma^4 - 456\Gamma^2 + 1073)}{1280} - \frac{\lambda}{z} \frac{(4\Gamma^2 - 9)(4\Gamma^2 - 61)}{192} \right. \\ & \left. \left. + \left(\frac{\lambda}{z} \right)^2 \frac{(4\Gamma^2 - 9)(4\Gamma^2 - 21)}{32} \right) + O(\hat{g}_s^6) \right]. \end{aligned} \quad (5.84)$$

We see that in this new expansion we again have worldsheets corresponding to a probe brane in a background quantified by Γ and perturbed by the dimensionless parameter

λ/z . Unlike the $\lambda = 0$ case, (5.38, 5.39), we see that Γ only appears in even powers and so we can view the physics as an FZZT brane probing an entirely closed string background. It would be very interesting to know why we apparently get ZZ boundary terms in (5.38, 5.39), and to understand the physics of that. After all, the background was purely closed before we put the FZZT brane in, and so surely the FZZT brane should still see a closed string background once it is added? A possible resolution to this problem is to consider the discussion of Section 5.2.2. There we saw that adding two FZZT branes is equivalent to adding one ZZ brane, and so an FZZT brane is in some way like half a ZZ brane. If rather than working with Γ in (5.38) we instead work with $C_+ = \frac{1}{2} + \Gamma$, we find that the expansion takes on a much simplified form (the analogous phenomenon occurs for (5.39)):

$$v(z) = -\frac{\nu C_+}{z} - \frac{\nu^3 C_+(C_+^2 - 1)}{z^4} - \frac{\nu^5(3C_+^5 - 13C_+^3 + 10C_+)}{z^7} + \dots \quad \text{as } z \rightarrow -\infty. \quad (5.85)$$

This expansion seems to be a lot simpler in terms of C_+ (the original Painlevé II parameter) than in terms of Γ . Notice also that only *odd* powers of C_+ appear in this expansion. A possible interpretation of this is that perhaps the FZZT brane has also joined the background of fluxes in some way. If we naively try to assign Euler numbers with boundaries to the worldsheets in this expansion then we would write $\chi = 2 - 2h - a - f$, where a is the number of powers of C_+ . However, we then notice that the total number of boundaries in each term, $a + f$, is always even. In the same way that an even number of boundaries could become R-R fluxes in the background $u(z)$ expansion, we could possibly conclude that the FZZT brane and ZZ branes together have become R-R fluxes in the $v(z)$ expansion. This would explain the appearance of C_+ as the natural expansion parameter, because the total charge of the background would then be $\Gamma + \frac{1}{2}$. Whether this interpretation is valid remains to be seen.

We can also make the same substitution for Γ in the large positive z expansion, (5.37). This yields:

$$v(z) = z^{1/2} + \frac{\nu C_+}{2z} - \frac{\nu^2(6C_+^2 + 1)}{16z^{5/2}} + \frac{\nu^3 C_+(16C_+^2 + 11)}{32z^4} + \dots \quad \text{as } z \rightarrow +\infty. \quad (5.86)$$

This expansion is again simpler in terms of C_+ than in terms of Γ , though we had a perfectly good interpretation before we made the substitution, so it is unclear if we

can learn anything from it. Due to the limited number of ways that C_+ appears¹⁶ it seems that we need no more than one FZZT boundary (i.e. $f = 1$) in each term to make a valid interpretation. This simplification remains to be understood. Let us note however, that once we turn λ back on we no longer find that the $\Gamma \mapsto C_+$ is a particularly natural substitution to make.

5.4 Summary and Discussion

In this chapter we have seen a number of convincing arguments for the quantisation of Γ . The most compelling of these came from the study of the explicit Bäcklund transformation (5.7), which we derived starting from the results of [56]. A solution of the string equation, $u(z)$, was found to be equivalent to a solution of the KdV hierarchy with $u(x, t) = t^{2\beta_m} u(xt^{\beta_m})$. Using this we showed that we could apply the original KdV Bäcklund transformation (2.58) to the string equations provided that we set $\lambda = 0$.

This led us to a study of the FZZT brane free energy, controlled by $v(z)$ and $\bar{v}(z)$. Once again, numerical results and the analysis of Bäcklund transformations indicate that Γ should be quantised. We showed that the background with one FZZT brane and Γ ZZ branes is equivalent to the background with one anti-FZZT brane and $\Gamma + 1$ ZZ branes, determining the charge of the FZZT brane to be half that of the ZZ branes. This tied in with the Bäcklund transformation itself, in that it gave a physical interpretation of its action.

We then analysed the functions $v(z)$ and $\bar{v}(z)$ at non-zero values of λ . We initiated the development of a simple set of rules at large positive z that could eventually allow the effects of λ to be added to the free energy without calculation. It is possible that once these rules are understood in more detail then it could have applications in other string theories with boundary cosmological constants.

In the next chapter we will again follow the principle of trying to apply known integrable systems results into minimal string theory. This time we will study the *negative* KdV hierarchy, which would correspond to the $(2, 4m)$ models with m negative. Since these models are not expected to exist as minimal conformal field theories then

¹⁶Even (odd) powers of C_+ only appear on worldsheets with odd (even) Euler number.

this will be a formal analytical continuation. From (3.15) we see that these string theories would naively have $\hat{c} > 9$; so if any physical predictions can be drawn from the models then this would be very interesting indeed.

Chapter 6

String Equations For The Negative KdV Hierarchy

So far we have studied the $(2, 4m)$ series of Type 0A string theories in great detail. We have seen that these correspond to string theories with \hat{c} given by:

$$\hat{c} = 1 - \frac{2(p-q)^2}{pq} = 5 - 4m - \frac{1}{m} \quad (6.1)$$

Since for $m > 1$ the models are non-unitary, we have seen that this series of models describes string theories living in one target spacetime dimension, the Liouville mode, φ . These models are known as *subcritical* because their dimensionalities lie below the critical value $D = 10$ ($\hat{c} = 9$). Whilst these theories are extremely interesting and tell us a lot about string theory in general, they nonetheless exist in a very uninteresting number of spacetime dimensions. It would therefore be very useful if we could extend the techniques of the previous chapter to more realistic dimensionalities.

One possible way to do this would be a formal analytical continuation, whereby m takes on other values besides the positive integers. Analytical continuations have proven to be very useful in the past¹, and so such an exercise would be fascinating to perform in the present case. Such a continuation of m has been speculated before, such as in [63] for instance, but never successfully carried out in the string theory context. However, there are two obvious problems.

Firstly, as we have seen, the (p, q) models only define minimal models for p and q positive coprime integers. So an extension outside this range would no longer define

¹Such as in defining the Riemann zeta function for instance. This provides one method of calculating the critical dimensionality of string theory.

a *known* minimal model. This theory would then presumably contain a non-finite number of primary operators. However, since string theories living in more realistic dimensions would presumably be more complicated anyway, then this may be a good thing.

Secondly, the hierarchy of string equations as formulated, (3.58), does not permit an obvious analytical continuation away from positive integral m . In the rest of this chapter we will see how to overcome this difficulty, at least as far as defining string theories for negative integer m (and zero).

Let us therefore assume that we can make such an analytical continuation. What dimensionalities do we expect for negative values of m ? Using the formula (6.1) we see that the string theories we would obtain would naively live in *greater* than ten spacetime dimensions. That is, they would be *supercritical*. For instance, $m = -1$ corresponds to $\hat{c} = 10$. Figure 6.1 illustrates this graphically by plotting \hat{c} against m for the $(2, 4m)$ models. Notice the ‘band gap’ that exists for $1 < \hat{c} < 9$. We have seen the difficulty in accessing this region via KPZ scaling before in (1.51).

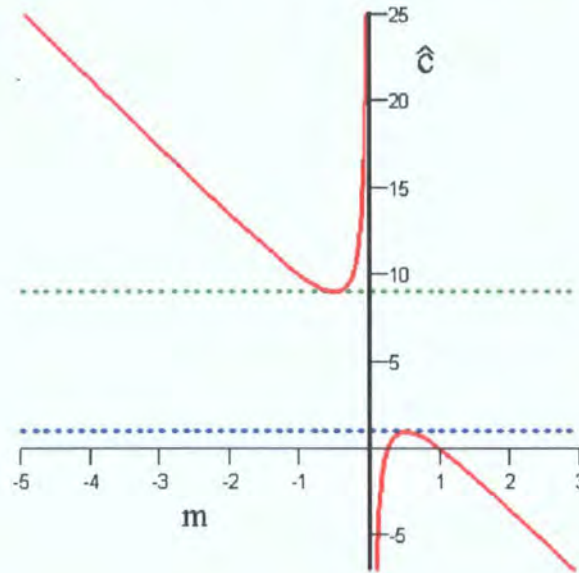


Figure 6.1: \hat{c} is plotted against m for the $(2, 4m)$ Type 0A string theory (red). Also shown are the lines $\hat{c} = 1$ (dotted blue) and $\hat{c} = 9$ (dotted green). By analytically continuing to negative values of m it may be possible to escape from the subcritical ($\hat{c} \leq 1$) region and break through to the supercritical ($\hat{c} > 9$) region.

In comparison to the subcritical models, relatively little is known about super-

critical string theories living in greater than the critical number of dimensions ($c > 25$, $\hat{c} > 9$). Some progress has been made however, such as in [64–70], but it would be very useful to learn more. Once again one finds that a solution to the background field equations is the linear dilaton. However, recall from Section 1.8 that the dilaton, Φ , of noncritical string theory is proportional to $Q\varphi$, with Q given by (3.17):

$$Q = \sqrt{\frac{9 - \hat{c}}{2}}. \quad (6.2)$$

So we see that for $\hat{c} > 9$ this quantity is imaginary, and hence the only way to define a sensible theory is to Wick rotate φ so that it becomes timelike instead of spacelike. This will lead to a theory that is either strongly coupled in the far past or strongly coupled in the far future. Consequently the theory is unstable, and it is not clear how to define an S-matrix. In an attempt to overcome some of these instabilities, other solutions to the background field equations have been suggested [71, 72], but there is still much to be understood.

Another problem with supercritical Type 0 theories is that the ‘tachyon’, which we recall was not actually a tachyon at all for $D \leq 2$, now *really is* a tachyon in the true sense of the word. In theories with tachyons, such as the bosonic string, one finds that perturbative quantities are often poorly defined. In the early days of string theory this was thought to be an insurmountable problem that rendered tachyonic theories catastrophically sick. However, more recently it has been realised that the presence of a tachyon indicates that one is expanding about a ‘false’ vacuum, which is unstable. In analogy to the Higgs mechanism of the Standard Model, the tachyons can often ‘condense’, leaving the theory at a stable vacuum without any tachyons. We will see a possible role for these ideas in the work below.

However, we should state from the outset that the main thrust of the work in this chapter is the mathematical structure itself, which is fascinating in its own right. Any physical interpretation of these models will be conjectural by its very nature, and so we will save such speculation until Section 6.6. We will see below that the string equations uncovered via this approach do yield apparently physically sensible results, at least perturbatively. Moreover, the models have some intriguing properties that may indeed relate them to supercritical theories in some way.

6.1 The String Equation Reformulated

We would like to find new string equations for the supposed $(2, -4|m|)$ models. As we have already noted, if we examine the form of the recursion relation (3.58) this does not seem feasible.

However, let us reformulate the KdV hierarchy in a way that will allow us to more easily define any analytic continuation. One can write the m -th member of the KdV hierarchy in the following form [10]:

$$\alpha_m \frac{\partial \tilde{u}(x, t_m)}{\partial t_m} = R'_{m+1}[\tilde{u}] \equiv K^m \cdot \tilde{u}', \quad K \equiv \partial^2 - 4\tilde{u} - 2\tilde{u}' \int_x, \quad (6.3)$$

where \int_x is an operator that integrates with respect to x . As we demonstrated in Chapter 5, one way to get from the KdV equation to the string equation is to search for scaling solutions of the former equation. To do this we again write $\tilde{u}(x, t_m) = t_m^{2\beta_m} u(z)$, with $z = xt_m^{\beta_m} \nu \equiv \tilde{z}\nu$. Substituting into (6.3) we find:

$$t_m^{2\beta_m-1} \alpha_m \beta_m (2u + \tilde{z}u') = t_m^{(2m+3)\beta_m} K^m \cdot u', \quad K \equiv d^2 - 4u - 2u'd^{-1}, \quad (6.4)$$

where d^{-1} refers to integration with respect to z (with an appropriate ν^{-1} factor). As in Chapter 5, we see that to eliminate t_m from this equation we need to choose $\beta_m = -1/(2m+1)$. We obtain:

$$\alpha_m \beta_m (2u + \tilde{z}u') = \tilde{\alpha}_m \nu K \cdot 1 = K^m \cdot u', \quad (6.5)$$

for some constant $\tilde{\alpha}_m$. Notice that we are always entitled to leave in an integration constant when we act with the operator K . It is clear that this will mix K^m with lower powers of K in any combination we wish. This allows us to modify (6.5) to:

$$\sum_{n=1}^m \tilde{t}_n K^n \cdot u' - K \cdot \nu = 0 \quad \Rightarrow \quad K \cdot \left(\sum_{n=0}^{m-1} \tilde{t}_{n+1} K^n \cdot u' - \nu \right) = K \cdot \mathcal{R}' = 0, \quad (6.6)$$

$$\mathcal{R}' \equiv \sum_{n=0}^{m-1} \tilde{t}_{n+1} K^n \cdot u' - \nu, \quad (6.7)$$

where the \tilde{t}_n are arbitrary constants. Comparison of (6.7) to (3.56) confirms that they are the same up to rescalings of the \tilde{t}_n . To proceed we now multiply (6.6) by \mathcal{R} and integrate. Upon doing this we once again obtain the string equation:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2. \quad (6.8)$$

6.2 The Negative KdV Hierarchy and its String Equations

As promised, we will now analytically continue the KdV equation to negative values of m . To do this is extremely simple: we just act on (6.3) by K^{-m} on both sides. This defines the *negative KdV hierarchy*, which is well known in the mathematical literature [73]. The first member of the series is related by a change of variables to the *Camassa-Holm* equation [73]. Using the above scaling we see that we should be able to derive the negative m analogues of the string equation. Using (6.5) we find that we can write:

$$K^{|m|+1} \cdot \nu = \kappa_{|m|} u', \quad (6.9)$$

for some constant $\kappa_{|m|}$. In general this will be an integro-differential equation rather than a mere differential equation, so obtaining solutions may be difficult. However in the case of the $m = -1$ model, a substitution will render it in the form of a pure differential equation. More generally we can again include integration constants in K . When we do this we find that the m -th equation gains contributions from the positive KdV hierarchy:

$$K^{|m|+1} \cdot \nu + \sum_{n=0}^{|m|-1} \hat{t}_n K^n \cdot u' = \kappa_{|m|} u'. \quad (6.10)$$

In the positive m models the constants \tilde{t}_n correspond to coefficients of operators in the worldsheet conformal field theory. It seems plausible that the \hat{t}_n play a similar role in the negative m models. It is therefore sensible to set the \hat{t}_n to zero for the time being and to try to understand the unperturbed models first. We will discuss what happens when they are non-zero later on. However, it is interesting to note that we can always choose a value of \hat{t}_0 that allows us to eliminate $\kappa_{|m|}$.

Before attempting any analysis of these new string equations, let us recall that they could potentially describe the supercritical $(2, -4|m|)$ models, if such models exist. If the equations yield any physically sensible results at all then this would be highly non-trivial. Should this be the case then there would remain the question of whether we have indeed broken through the $\hat{c} = 1$ barrier and reached $\hat{c} > 9$; or whether we have bounced off and are now studying some other $\hat{c} < 1$ model? We will leave these

questions until Section 6.6. Regardless of whether we are dealing with supercritical models or not, we will still refer to them as such in what follows, to distinguish them from the earlier subcritical models.

We will later write the negative KdV string equations in their natural form (6.8), but for now let us analyse (6.9) in its present form. We will see that this will yield valuable information that would otherwise have been obscured. The first member of the hierarchy is $m = -1$. By writing $u(z) = w'(z)$ we obtain the following differential equation:

$$4w^{(3)} + \tilde{z}w^{(4)} - 8(w')^2 - 6\tilde{z}w'w'' - 2ww'' = \kappa_1 w''. \quad (6.11)$$

If this equation describes the $(2, -4)$ superconformal model as our analytic continuation suggests, then it would correspond to $\hat{c} = 10$. So this would be eleven dimensional Type 0A superstring theory, with ten ‘ordinary’ dimensions plus the timelike Liouville mode. The first observation of note is that the constant κ_1 can be eliminated by adding a constant term to $w(z)$. This term will be an analytic contribution to the free energy and as such is non-universal so can be ignored. So we can set $\kappa_1 = 0$ without loss of generality.

Recalling the subcritical models, we should naturally expect the leading order behaviour of $u(z)$ to be $z^{1/m}$ for the $(2, 4m)$ model. So for $m = -1$ we expect $w(z)$ to lead with a $\ln|z|$ contribution. This has a problematic interpretation and leads to terms of the form $z^a \ln|z|$ in the free energy. It turns out that the $\ln|z|$ ansatz is not a solution of the equation anyway, so we are saved from having to make such an interpretation. So on first glance it looks as if this cannot possibly be the $(2, -4)$ model. This is not necessarily the case however, and to show this we need to recall the origin of the the expected $1/m$ power (c.f. Section 3.7). This comes from the KPZ scaling described in Section 1.8. Recall that, in the case where z is the bulk cosmological constant, one expects the first term in $u(z)$ to lead like $z^{-\gamma_{str}}$, where γ_{str} is given by:

$$\gamma_{str} = 2 + \frac{1}{12} \left(c - 25 - \sqrt{(25 - c)(1 - c)} \right), \quad (6.12)$$

$$\gamma_{str} = 2 + \frac{1}{4} \left(\hat{c} - 9 - \sqrt{(9 - \hat{c})(1 - \hat{c})} \right), \quad (6.13)$$

where we use the c formula in the bosonic theory, and the \hat{c} formula in the Type

0A theory². Recall that in non-unitary theories ($c < 0$), z controls the operator of most negative dimension in the conformal field theory [5], rather than being the bulk cosmological constant itself, and so KPZ scaling does not work in the same way. It was this argument that originally led to the conclusion that the m -th multicritical point corresponded to the $(2, 2m - 1)$ model, and not the $(m, m + 1)$ model as was originally believed. However, since we expect the string equations of the negative KdV hierarchy to have $\hat{c} > 0$, then we also expect them to be unitary. So perhaps KPZ scaling should work after all? If this is true then we would expect (6.11) to have a solution with $z^{-1/2}$ leading order behaviour for $w(z)$. This is not the case though, and so whether this is a fundamental problem for the new string equations (6.9) is unclear. However, the KPZ results of (6.12) and (6.13) clearly make no sense for the $1 < c < 25$ and $1 < \hat{c} < 9$ regimes, so it is at least plausible that they do not hold exactly in the supercritical regime either, possibly because the Liouville mode is timelike [66]. We will take this optimistic approach and continue with our analysis of (6.11).

We proceed by looking for a leading order solution of the form $w(z) \sim Bz^b$, for some power b and coefficient B . Substituting into (6.11) we find that there are two types of contribution: terms proportional to z^{b-3} , and those proportional to z^{2b-2} . The Bz^b ansatz will be appropriate for most equations that are similar in form to (6.11). Accordingly we will explain how to proceed in this general case, using the specific example of (6.11) to illustrate matters. In most cases there will be two ways to fix a Bz^b term in the perturbative expansion; and we will call these *power determined* and *coefficient determined*.

The first involves simply solving for when two or more terms become equally dominant. In the specific case of (6.11) this means that we solve the equation $b - 3 = 2b - 2$. In the general case one can plot all the contributing terms as lines on a graph, with b on the horizontal axis, and the powers of z on the vertical axis. One then looks for dominant crossing points on the graph by starting on the top line at $b = -\infty$ and tracing down until one reaches an intersection of two or more lines. One can continue in this way, always staying on the highest line until $b = +\infty$ is reached. Once all

²The first of these results is as in Chapter 1, the second we are introducing for the first time from [3, 16].

the critical values of b have been power determined in such a way it is possible to determine the corresponding values of B . We call this power determined because b depends only on the general form of the differential equation, not the numerical factors weighting the different terms themselves. This method was employed exclusively in calculating the expansions of the previous chapters, and we earlier employed it without explanation.

If two or more terms have the same power in an equation then one may find b another way: the terms form a subgroup that vanishes if b takes on special values. This will be true for all values of B , and hence corresponds to the introduction of an arbitrary constant into the solution. We call this coefficient determined, because b will depend on the numerical coefficients in the differential equation, and not just the general form of that equation.

In the case of (6.11) we find that $b = -1$ is the only power determined value. For $b < -1$ the z^{b-3} terms will dominate; for $b > -1$ the z^{2b-2} terms will dominate. The z^{b-3} subgroup will vanish if b satisfies the following equation:

$$b(b-1)(b-2)[4+(b-3)] = 0. \quad (6.14)$$

So it will vanish for $b = -1, 0, 1, 2$. However, all these values apart from the first lead to subleading terms ($b > -1$) and so can be discounted. The z^{2b-2} subgroup vanishes if:

$$8b^2 + 6b^2(b-1) - 2b(b-1) = 0. \quad (6.15)$$

So it will vanish for $b = -1, 0, 1/3$. The $b = -1$ term is common to both the z^{b-3} and z^{2b-2} subgroups and hence will override the earlier power determined value to give us an arbitrary coefficient of a z^{-1} term. We will return to this later. The $b = 0$ solution is just the trivial constant that we eliminated before. So the only new value is $b = 1/3$. We write $w(z) = A^2 z^{1/3}/(3\nu) + w(z)$ for some free parameter A (why we have written it in this way will become clear shortly). We then solve the resulting equation in the same way for the new $w(z)$. There is an arbitrary constant at the next order too. We will call it Γ . The first few terms of the solution are:

$$w = \frac{A^2 z^{1/3}}{3\nu} + \frac{2A\Gamma}{3z^{1/3}} - \frac{\nu(12\Gamma^2 - 5)}{36z} + \frac{2\nu^2\Gamma(\Gamma^2 - 1)}{9Az^{5/3}} - \frac{\nu^3\Gamma^2(\Gamma^2 - 1)}{9A^2z^{7/3}} + \dots \quad (6.16)$$

Integrating up once (and dividing by ν) to get the free energy, we find:

$$F = \frac{A^2 z^{4/3}}{4\nu^2} + \frac{A\Gamma z^{2/3}}{\nu} - \frac{(12\Gamma^2 - 5)}{36} \ln|z| - \frac{\nu\Gamma(\Gamma^2 - 1)}{3Az^{2/3}} + \frac{\nu^2\Gamma^2(\Gamma^2 - 1)}{12A^2z^{4/3}} + \dots \quad (6.17)$$

We see that once again we can define a dimensionless parameter $g_s = \nu/(Az^{2/3})$, though this time it contains our constant A . We can therefore write the free energy as:

$$F = \frac{1}{4}g_s^{-2} + \Gamma g_s^{-1} - \frac{(12\Gamma^2 - 5)}{36}g_s^0 \ln|z| - \frac{\Gamma(\Gamma^2 - 1)}{3}g_s + \frac{\Gamma^2(\Gamma^2 - 1)}{12}g_s^2 + \dots \quad (6.18)$$

The interpretation of these terms can again be made in terms of worldsheets with some number of handles and boundaries. Γ would then represent the number of a species of background branes. Is it the same Γ that we found in the positive m hierarchy? Let us evaluate the Bäcklund transformation (5.14). We should ask whether we expect the Bäcklund transformation to hold at all in these new models. The answer is yes, for the following reason. The Lax equations for the first member of the negative KdV hierarchy are given by [73]:

$$[\partial^2 - u(x, t)] \psi = \lambda \psi, \quad w \frac{\partial \psi}{\partial x} - \frac{1}{2} \frac{\partial w}{\partial x} \psi = 2\lambda \frac{\partial \psi}{\partial t}, \quad u = \frac{\partial w}{\partial x}. \quad (6.19)$$

The first of these equations is identical to (2.33), implying that we can once again derive the Miura map (2.27). The Bäcklund transformation (5.14) is derived from the Miura map by noticing that if the mKdV hierarchy has a solution $v(z)$ then it also has a solution $-v(z)$. The mKdV hierarchy can be formulated in a similar manner to (6.3). If we do this then it becomes clear that the property in question should still be true for the negative mKdV hierarchy. So we expect that the Bäcklund transformation should still be relevant in our new models. Plugging the solution (6.16) into (5.14), we find that the transformed expansion is indeed of the same form as (6.16). What is more, Γ has indeed changed by an integer. This is evidence that the model may be describing a valid physical theory.

Notice also that the solution (6.16) appears to have special properties if $\Gamma = 0, \pm 1$. We calculated the expansion up to tenth order and found that, after the torus term, all the higher order contributions are proportional to $\Gamma(\Gamma^2 - 1)$ multiplied by some other polynomial in Γ . This means that they vanish at these special values of Γ . As alluded to above, there is also a second large- z solution to (6.11) that leads with $w(z) \sim Cz^{-1}$, where C is a constant. It turns out that the second term in the expansion is also coefficient determined. What is more, the condition for the relevant subgroup to vanish now contains C itself. So the powers appearing in the series actually depend on our choice of the leading term. This is something that has no

analogue in the subcritical models. We find that if the next leading order term is of the form z^b then b must satisfy:

$$b = 1 \pm \sqrt{1 - 4C}. \quad (6.20)$$

Looking back at (3.69), we see that the first term in that expansion also contributes a z^{-1} contribution to $w(z)$. In fact, for the subcritical models this first term is shared by all the string equations in the hierarchy. The coefficient is the same too. So let us speculate that this first term is shared by all the members of the negative hierarchy as well. Writing $C = -(4\Gamma^2 - 1)/4$ we find that b simplifies to give $b = 1 \pm 2\Gamma$. Assuming for now that Γ is positive and greater than unity, we take the negative sign choice in this equation for b . We then find that the following perturbative solution holds at least up to tenth order:

$$w = -\frac{\nu(4\Gamma^2 - 1)}{4z} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \nu^{n+1} B_1^n}{4^{n-1} (\Gamma - 1)^{n-1}} z^{2n-1-2n\Gamma}, \quad (6.21)$$

$$F = -\frac{4\Gamma^2 - 1}{4} \ln |z| + \sum_{n=1}^{\infty} \frac{(-1)^n \nu^n B_1^n}{2^{2n-1} n (\Gamma - 1)^n} z^{-2n(\Gamma-1)}. \quad (6.22)$$

Here B_1 is another constant. Assuming that the dimensionless string coupling is of the same form, $g_s = \nu/(Az^{2/3})$, as in (6.18),³ we can rewrite the above free energy in the following form:

$$F = -\frac{4\Gamma^2 - 1}{4} \ln |z| + 2 \sum_{n=1}^{\infty} \frac{\tilde{B}_1^n}{n} g_s^{3n(\Gamma-1)}, \quad (6.23)$$

where we have absorbed all extra Γ and ν dependence into the new constant \tilde{B}_1 . The physical interpretation of this series is indeed curious. The surfaces contributing to the free energy depend on the value of Γ . What is more, it seems that we must have $\Gamma > 1$ to have \tilde{B}_1 non-zero⁴. It is also intriguing that we seem to have obtained a general form for the series to all orders in perturbation theory. Many of these mysteries are apparently solved by the Bäcklund transformation, under which it turns out that one must set $\tilde{B}_1 = 0$ to make Γ change by an integer. If we take the Bäcklund transformation as in some way sacrosanct to the physics of the theory, then it seems that we must conclude that the physical theory will always have $\tilde{B}_1 = 0$, thus eliminating

³Though there is no conclusive reason why this needs to be the case.

⁴Recall that we chose $b = 1 - 2\Gamma$. If instead we choose $b = 1 + 2\Gamma$ then we can have $\Gamma < -1$.

the need to explain this apparently troublesome physics. The full story is actually more complicated than this, and we will reexamine expansions with Γ dependent coefficients in Section 6.5, where we will learn a lot more about them. Until then we will often ignore them and instead concentrate only on the more interesting solutions that are analogous to (6.16).

Note also, that unlike in the $m = 1$ model discussed earlier, both of the possible large- z expansions, (6.16) and (6.21) (with $\tilde{B}_1 = 0$), can be real at both large negative and large positive z . So it is perhaps possible that we can have (6.16) or (6.21) being valid in both weak coupling regimes. We will see a useful interpretation of this a little later. Unfortunately we have not yet been able to solve the string equation of (6.11) numerically. If this could be done then it would yield vital information about which boundary conditions we should choose. For now let us consider the Miura map (2.27), and calculate what we would naively conclude are the FZZT free energies corresponding to (6.16) and (6.21). We could leave $\tilde{B}_1 \neq 0$ in the latter case, but this is not very insightful. For (6.16) we again find two solutions $v(z)$ and $\bar{v}(z)$:

$$\begin{aligned} v &= \frac{A}{3z^{1/3}} - \frac{\nu(2\Gamma+1)}{6z} + \frac{\nu^2\Gamma(\Gamma+1)}{3Az^{5/3}} - \frac{\nu^3\Gamma(2\Gamma^2+3\Gamma+1)}{9A^2z^{7/3}} + \dots, \\ \bar{v} &= -\frac{A}{3z^{1/3}} + \frac{\nu(2\Gamma-1)}{6z} - \frac{\nu^2\Gamma(\Gamma-1)}{3Az^{5/3}} + \frac{\nu^3\Gamma(2\Gamma^2-3\Gamma+1)}{9A^2z^{7/3}} + \dots. \end{aligned} \quad (6.24)$$

Turning to the solutions of the Miura map associated with (6.21) we find:

$$v = -\frac{\nu(1+2\Gamma)}{2z}, \quad \bar{v} = -\frac{\nu(1-2\Gamma)}{2z}, \quad (6.25)$$

which are exact solutions. Once again we can interpret these results in terms of open strings having f boundaries ending on a new type of brane. Notice that these $v(z)$ and $\bar{v}(z)$ expansions respect the $(-1)^f$ symmetry of the subcritical models. If these models are indeed related to supercritical string theories then it is questionable whether we should expect to find the physics given in terms of ZZ and FZZT branes like in the subcritical models. Here we merely point out what the mathematics is telling us. We will postpone discussion of the physics until Section 6.6.

6.3 Higher String Equations

Let us study the next string equation in the hierarchy, $K^3 \cdot \nu = 0$, which corresponds to the supposed $(2, -8)$ model. It is an integro-differential equation with the naive value of \hat{c} equal to $27/2$:

$$6w^{(5)} + \tilde{z}w^{(6)} - 36(w'')^2 - 48w'w^{(3)} - 20\tilde{z}w''w^{(3)} - 10\tilde{z}w'w^{(4)} - 2ww^{(4)} + 32(w')^3 + 24\tilde{z}(w')^2w'' + 8ww'w'' + 4w'' \int d\tilde{z} (4(w')^2 + 3\tilde{z}w'w'' + ww'') = \kappa_2 w''. \quad (6.26)$$

With κ_2 non-zero it appears that there are no perturbative solutions to this equation at all. So it seems as if we are forced to set the $\kappa_{|m|}$ to zero to proceed. When we do this we realise that the negative KdV string equations have the form $K^{|m|+1} \cdot \nu = 0$. So a solution of the $|m|$ -th model will also be a solution of the $(|m| + 1)$ -th. There is one new solution to (6.26) though⁵:

$$w = \frac{A^2 z^{3/5}}{15\nu} + \frac{2A\Gamma}{5z^{1/5}} - \frac{\nu(12\Gamma^2 - 7)}{60z} + \frac{2\nu^2\Gamma(2\Gamma^2 - 3)}{15Az^{9/5}} - \frac{\nu^3(18\Gamma^4 - 42\Gamma^2 + 7)}{45A^2z^{13/5}} + \dots \quad (6.27)$$

To this expansion we can assign the dimensionless string coupling $g_s = \nu/(Az^{4/5})$. Once again, the parameter Γ behaves as expected under the Bäcklund transformation (5.14). In general it seems that the $|m|$ -th string equation of the negative KdV hierarchy has a solution $u(z)$ with $z^{-2/(2|m|+1)}$ leading order behaviour. This is in addition to it having all the solutions of the lower equations. Notice that we can also define the $m = 0$ model in (6.9). It is to this model that the $u(z) = (4\Gamma^2 - 1)/(4z^2)$ solution most naturally belongs. Unfortunately, we see from (6.1) that the naive value of the central charge for this solution is infinite, and it also has a pole at $z = 0$. This will be discussed further in Section 6.6.

Since all the perturbative solutions are valid at both negative and positive large- z , we seem to have a lot of choices of boundary conditions with no obvious way to choose between them. However, so far we have been dealing with the differentiated analogue of the string equation (5.8). That is, we have been studying $K \cdot \mathcal{R}' = 0$. This we have done for reasons that will become clear below. A formulation of the string equations analogous to (3.67) is possible however, and we will derive it now. This will clarify a

⁵Once again ignoring solutions with Γ dependent coefficients.

few issues and lead to some important restrictions upon the solutions we have already obtained.

6.4 The Full String Equation

Recall our basic string equation (6.9):

$$K^{|m|+1} \cdot \nu = \kappa_{|m|} u'. \quad (6.28)$$

We can integrate up to get the analogue of (3.67) if we can write it in the form $K \cdot \mathcal{R}' = 0$. We see that if $\kappa_{|m|} = 0$ then we can achieve this in closed-form:

$$K \cdot (K^{|m|} \cdot \nu) = 0 \quad \Rightarrow \quad \mathcal{R}' = K^{|m|} \cdot \nu. \quad (6.29)$$

Earlier it was explained how leaving $\kappa_{|m|}$ non-zero caused problems when searching for perturbative solutions. We also saw how it could always be absorbed into \hat{t}_0 in (6.10). So although it may be possible to leave $\kappa_{|m|}$ non-zero and still compute an appropriate \mathcal{R} , it seems reasonably natural to set it to zero. This done, \mathcal{R} satisfies the following equation:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Lambda^2, \quad (6.30)$$

which is the usual string equation of the positive m models as we had hoped. Λ is a constant that we suspect will be related to Γ . For the $m = -1$ model we find that \mathcal{R} is given by:

$$\mathcal{R} = -2\nu [w(z) + \tilde{z}w'(z)], \quad (6.31)$$

which yields the full string equation:

$$\begin{aligned} & w^2w' + 2\tilde{z}w(w')^2 + \tilde{z}^2(w')^3 - \frac{3}{2}ww'' - \frac{1}{2}\tilde{z}ww^{(3)} \\ & - \frac{1}{2}\tilde{z}w'w'' - \frac{1}{2}\tilde{z}^2w'w^{(3)} + (w')^2 + \frac{1}{4}\tilde{z}^2(w'')^2 - \frac{\Lambda^2}{4} = 0. \end{aligned} \quad (6.32)$$

We can substitute our original perturbative solutions into this equation to see if they are still valid. Starting with (6.16) we uncover the following, perhaps surprising, result: the expansion actually satisfies (6.32) for *all* values of Γ . So it seems that

Γ and Λ are independent parameters. Instead we find that it is actually A that is related to Λ :

$$\nu\Lambda = \pm \left(\frac{2A}{3}\right)^3. \quad (6.33)$$

Analogous relations can be found in the other members of the hierarchy⁶. So it has transpired that our constants A and Λ are not independent in the context of (6.30). We also see that Γ does not appear explicitly in either version of the string equation (though for $m = -1$ we will show below how it can indeed be obtained as an integration constant), but instead it can always be found via the method of coefficient determination and the use of the Bäcklund transformation (5.14). This is why we studied the differentiated equation first: it is much easier to identify the correct physical Γ .

However, in this framework the expansion given by (6.16) is clearly singular if $\Lambda = 0$ (because $A = 0$). Moreover, the expansion given by (6.21) is not a solution of (6.32) *unless* $\Lambda = 0$. The same appears to be true of the $|m|$ -th model: for $\Lambda \neq 0$ the theory is forced to be in its ‘natural’ state with $u(z) \sim z^{-2/(2|m|+1)}$ boundary conditions in both weak coupling regimes; but when $\Lambda = 0$ it must drop down to a solution of a lower member of the hierarchy. This is because if Λ is explicitly zero in (6.30) then we can choose $\mathcal{R} \equiv \mathcal{R}_{|m|} = 0$, implying that the solutions to the differentiated $K \cdot \mathcal{R}'_{|m|-1} = 0$ equation of the *next lowest model* will hold. When the theory does drop out of its natural state there seems to be no obvious way of determining which of the lower solutions it will end up at; or what its new values of Λ and Γ will be when it gets there.

It turns out that for $m = -1$ we can actually find Γ as an integration constant. To do this we start with the differentiated string equation, (6.11), and multiply it by z . It then becomes a total derivative and (ignoring the κ_1 term) integrating we find:

$$w^2 - 3\tilde{z} (w')^2 - 2zww' - 2w' + 2\tilde{z}w'' + \tilde{z}^2w^{(3)} = 2\nu^2\Lambda\Gamma \quad (6.34)$$

Unfortunately, we have not yet managed to perform a similar integration to obtain a generalised form of this equation involving $\mathcal{R}[u]$ for all m . The multiplying by z trick seems to work for $m = 0$, but not obviously for $m < -1$.

⁶The exception being the $m = 0$ model, which has $\mathcal{R} = z$. In this model Γ and Λ are the same parameter.

Finally in this section, let us determine whether the explicit form of the Bäcklund transformation, (5.7), holds now that we have correctly identified $\mathcal{R}[u]$. It turns out that it does, although one must change each appearance of Γ to a Λ :

$$u_{\Gamma \pm 1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\Lambda\mathcal{R}' + 4\nu^2\Lambda^2}{4\mathcal{R}^2}. \quad (6.35)$$

This is another nice result that ties matters in with the subcritical case.

6.5 Pole Solutions and The Role of The \hat{t}_n .

In (6.21) we found a solution to the $m = -1$ string equation that had Γ dependent powers (with a $z^{1-2\Gamma}$ next-to-leading order term⁷), but no obvious physical interpretation. This solution had a parameter B_1 that we could turn on for $\Gamma > 1$, but were forced to set to zero in order for the Bäcklund transformation to change Γ by an integer. The $m = -2$ string equation also has this solution; and also another that takes a similar form, leading with $-\nu(4\Gamma^2 - 1)/(4z)$, but having a next-to-leading order term proportional to $z^{3-2\Gamma}$, meaning that we must have $\Gamma > 2$ to turn this term on. In addition to this, the $m = -2$ solution is marked by the appearance of a second arbitrary constant at order $z^{1-2\Gamma}$. For instance, in the case of $\Gamma = 4$ we have:

$$w(x) = \frac{-63\nu}{4z} + \frac{B_2\nu^3}{z^5} + \frac{B_1\nu^4}{z^7} - \frac{7\nu^5 B_2^2}{48z^9} - \frac{5\nu^6 B_1 B_2}{24z^{11}} + \frac{\nu^7(15B_2^3 - 64B_1^2)}{768z^{13}} + \dots \quad (6.36)$$

Note that in the $B_2 \rightarrow 0$ limit we recover the $m = -1$ solution with $\Gamma = 4$. Moving on to $m = -3$, we find a solution with three constants, B_1 , B_2 and B_3 , which turns out to be contain the two lower solutions as limits. As one would now expect, at the $|m|$ -th level one finds a solution, $F_\Gamma^{(|m|)}$, which has $|m|$ arbitrary constants within it and must have $\Gamma > |m|$ to exist. Limiting cases of $F_\Gamma^{(|m|)}$ lead to all of the $F_\Gamma^{(n)}$, $n = 0, \dots, |m|$.

As we have tried to do before, it would be tempting to interpret these perturbative series as string genus expansions. This would most likely involve closed strings only, and the free parameters would then correspond to various backgrounds, possibly involving fluxes. Before jumping to any premature conclusions, let us first examine the effects of the Bäcklund transformation again, because from our $m = -1$ experience we might expect that we must set the B_n to zero.

⁷We are going to assume $\Gamma \geq 0$ for clarity.

For comparison with the present case, let us define (6.16) as $E_\Gamma^{(1)}$, (6.27) as $E_\Gamma^{(2)}$, and so on and so forth. We know from earlier results that the Bäcklund transformation acts on these solutions as:

$$\mathbb{B} \left(E_\Gamma^{(|m|)} \right) = \mathbb{B} \left(E_{\Gamma \pm 1}^{(|m|)} \right), \quad (6.37)$$

where we have introduced an operator \mathbb{B} to denote the action of the Bäcklund transformation (as in Chapter 2). For the $F_\Gamma^{(|m|)}$ solutions it turns out that we instead have:

$$\mathbb{B} \left(F_\Gamma^{(|m|)} \right) = \mathbb{B} \left(F_{\Gamma \pm 1}^{(|m| \pm 1)} \right). \quad (6.38)$$

So the solution associated with any given model is actually connected with the solution of the neighbouring model via the Bäcklund transformation. Recall that in order to define the m -th equation of the negative KdV hierarchy, we had to multiply both sides of (6.3) by K^{-m} , which yielded $K^{|m|+1} \cdot \nu = 0$ (since $\kappa_{|m|} = 0$). Equally, if this is acceptable then we could instead have multiplied both sides by K^{-m+1} to give $K^{|m|+2} \cdot \nu$, which is the $(|m| + 1)$ -th string equation. If this procedure is allowed then we see that there is a fundamental ambiguity in the definition of which model we are studying at any particular time. The Bäcklund transformation (6.38) appears to verify this ambiguity, because it seems to be telling us that we should consider any solution to be a solution of the hierarchy itself, rather than a solution to any particular equation in the hierarchy.

We see from (6.38) that the action of the Bäcklund transformation appears to generate arbitrary constants. Recall also that we have also seen this in Chapter 5 (see the discussion around (5.19)), where we found that the Bäcklund transformation created a perturbative parameter at large negative z . There we had to set this parameter to zero, because otherwise the Bäcklund transformed expansion was not a solution to the string equation. In the present case we see that the Bäcklund transformed functions are solutions of a string equation, just not the one we started with. However, we have so-far not been able to formulate the corresponding ‘time-part’ of the transformation. If we could do this then we may be forced to set some of the arbitrary constants to zero. This would make sense because a limit of $F_{\Gamma+1}^{(|m|+1)}$ is $F_\Gamma^{(|m|)}$, in which case (6.38) would represent a transformation between solutions of the same equation, thus relieving some of the ambiguity.

Continuing this train of thought, we realise that all the $F_\Gamma^{(|m|)}$ can be generated from the fundamental set $F_\Gamma^{(0)}$, which corresponds to the $u(z) = \nu(4\Gamma^2 - 1)/(4z^2)$ exact solution. Since these initial solutions are so simple, it is tempting to try and solve the Bäcklund transformation *exactly*, rather than perturbatively. In general we use MAPLE to do this. For $F_\Gamma^{(1)}$ we find (ignoring factors of ν from now on for clarity):

$$F_\Gamma^{(1)} = -\frac{z^{2\Gamma-2}(2\Gamma+1)(2\Gamma-1) - B_1(2\Gamma-3)(2\Gamma-5)}{4z(z^{2\Gamma-2} - B_1)}, \quad (6.39)$$

where B_1 is again an arbitrary constant (but not necessarily the same B_1 we encountered before). Again we must have $\Gamma > 1$ for this to be valid⁸. Note that in the case $B_1 = 0$ we recover the $m = 0$ solution as we would expect. If we expand (6.39) at large z then we find that it does indeed give the result displayed in (6.21). However, we can now take the limit $B_1 \rightarrow \infty$, which is non-perturbative. We see when we do this that we obtain $w(z) = -(4(\Gamma-2)^2 - 1)/(4z)$. So this is just $F_{\Gamma-2}^{(0)}$, which is again a manifestation of the Bäcklund transformation creating a kink-like solution (see Chapter 5), with the parameter B_1 interpolating between $F_{\Gamma-2}^{(0)}$ and $F_\Gamma^{(0)}$. This is natural because we obtained the solution by starting from $F_{\Gamma-1}^{(0)}$. This again fits with the idea of the Bäcklund transformation creating and destroying solitons. The difference between here and Chapter 5 is that this time the full Bäcklund transformed functions *are* allowed by the string equations.

For higher $|m|$ we find a similar story. For instance, taking two Bäcklund transformations of $F_\Gamma^{(0)}$, leads to $F_{\Gamma+2}^{(2)}$ with two free parameters. The ultimate limits of $F_{\Gamma+2}^{(2)}$ are then $F_{\Gamma-2}^{(0)}$, $F_\Gamma^{(0)}$ and $F_{\Gamma+2}^{(0)}$.

So it seems that we have uncovered a whole series of soliton-like solutions within the hierarchy. The problem comes when we try to interpret them in string theory terms. Unfortunately we see that they all have poles, and so whether they would be physically allowable is questionable. We will discuss this in the next section. However, we have not yet turned on the \hat{t}_n coefficients in (6.10). It is possible that they will play a crucial role in our understanding, so we will study them now.

⁸Actually, the Bäcklund transformation allows us to obtain a solution for $\Gamma = 1$, but this involves $\ln|z|$ terms and hence was not discovered perturbatively. It turns out that we can also obtain exact solutions for imaginary values of Γ also. Similar solutions exist in the higher $|m|$ models.

In the case of the subcritical models, the \tilde{t}_n in (6.6) represent the coefficients of other operators in the worldsheet conformal field theory and, because the theories are minimal, there are a finite number of these. The effect of turning them on is to perturb the asymptotic expansions in some non-trivial way. It is therefore natural to guess that they will play a similar role in the negative KdV string equations.

This is certainly not the case however. Firstly, it turns out to be possible to absorb the effects of $\hat{t}_{|m|-1}$ into a non-universal constant in $w(z)$. This is the same effect we witnessed in the $m = -1$ model where κ_1 was absorbed in this way. As was hinted at earlier, turning on the lower \hat{t}_n is incompatible with the $E_\Gamma^{(|m|)}$ solutions. They only exist for $\hat{t}_n = 0$ ($n < |m| - 1$). This contrasts with the $F_\Gamma^{(|m|)}$ solutions, which only exist if $\Lambda = 0$.

It is when we consider the $F_\Gamma^{(|m|)}$ solutions that the \hat{t}_n begin to have non-trivial effects. Instead of perturbing the original solutions like in the subcritical models, it seems that the \hat{t}_n merely act to single out some of the existing solutions. The \hat{t}_n coefficients themselves never appear explicitly in the expansions at all.

For $m = -1$ there are no non-trivial \hat{t}_n coefficients. In the case of $m = -2$ we can turn on \hat{t}_0 , though the only solution compatible with this is $w(z) = 0$. In fact, turning on \hat{t}_0 in any model only permits $w(z) = 0$. At $m = -3$ we can also turn on \hat{t}_1 . Turning it on by itself⁹ permits two solutions, one of which is $w(z) = 0$ again. The second solution is $F_\Gamma^{(1)}$, but *only* for $\Gamma = 3/2$. For $m = -4$, turning on \hat{t}_1 again permits $F_{3/2}^{(1)}$ and $w(z) = 0$; whereas turning on \hat{t}_2 by itself allows both these *and* $F_{5/2}^{(2)}$. This pattern continues in all the higher models. Interestingly, all these solutions are those that are connected to the ‘nothing-at-all’ solution, $F_{1/2}^{(0)} = 0$, by the Bäcklund transformation.

6.6 Summary and Discussion

In the preceding sections we have seen that it is possible to define a family of string equations associated with the negative KdV hierarchy. It is encouraging that the perturbative expansions obtained from these string equations can again be interpreted sensibly in terms of string worldsheets with various numbers of handles and bound-

⁹In general the effects of the lowest \hat{t}_n are dominant, so turning on \hat{t}_3 will override the effects of \hat{t}_5 for instance.

aries. What is more, it seems that we can once again identify a parameter Γ that changes by an integer under the Bäcklund transformation (2.58). It is of course very possible that we have merely uncovered a mathematical structure that is mimicking sensible physics. However, it is interesting that Γ still plays the same role as in the positive KdV string equations, yet no longer appears explicitly in the string equation (6.30).

Assuming for now that the models do describe physical theories, there remains the question of what these theories actually are. Our naive expectation has always been that they are the supercritical $(2, -4|m|)$ models, with $\hat{c} = 10, 27/2, 52/3, \dots$ ($m = -1, -2, -3, \dots$ respectively). We see from (6.1) that, for $m \lesssim -2$, \hat{c} is well-approximated by $4|m| + 5$, and hence the dimensionality increases almost linearly with $|m|$. However, we have already seen above that the notion of which model we are dealing with at any one time is slightly ambiguous. Furthermore, as we have seen, supercritical Type 0 strings have physical closed string spacetime tachyons. This usually leads to difficulties in defining amplitudes, and so the fact that we have obtained sensible perturbative expansions for the $E_\Gamma^{(|m|)}$ may indicate that the renormalised theory has (at least partially) overcome this problem. The $F_\Gamma^{(|m|)}$ solutions certainly have poles in their free energies, and this is an indication that these solutions are non-perturbatively sick, possibly because of the tachyon's effects. It is unclear whether the $E_\Gamma^{(|m|)}$ also have poles or not. As was mentioned earlier, the presence of a physical tachyon indicates that we are expanding about a 'false' vacuum. It is known that these tachyons can condense, leaving the theory at a more sensible vacuum. If the $E_\Gamma^{(|m|)}$ are non-perturbatively smooth then this may be what has happened. We will see another possible interpretation involving tachyon condensation below.

However, if the hierarchy really is describing supercritical string theories, then it seems strange that we are again seeing the physics described in terms of objects that look very similar to the ZZ and FZZT branes of the subcritical models. In supercritical string theory the Liouville direction is timelike and so the locations of such branes become problematic. Nevertheless, some progress has been made on these issues, such as in [74–76], where results from the usual spacelike Liouville theory are analytically continued to obtain sensible answers. Accordingly, one could perhaps interpret the

branes we are seeing in terms of S-branes or D-instantons [77].¹⁰ So if the negative KdV hierarchy is indeed describing some aspect of supercritical string theories, then perhaps we should not be too surprised that we are again seeing what look like ZZ and FZZT branes.

One would also expect, perhaps naively, that supercritical string theories would be far richer in nature than their subcritical counterparts, possibly involving higher dimensional ZZ and FZZT branes. Yet the negative KdV models do not appear to be a great deal more complicated than those of the positive hierarchy. Perhaps this means that each model only describes a certain subsector of a supercritical string theory, rather than the whole theory itself. Another possibility is that we are instead dealing with some new family of subcritical models. We should always be careful however, because we are lacking an obvious target spacetime interpretation. Until such an interpretation can be made it is hard to draw any firm conclusions as to the nature of the physical theory.

Recall that each member of the subcritical ($m > 0$) series of $(2, 4|m|)$ string equations has two large- z solutions. One of these corresponds to a weak coupling regime with D-branes, the other to a regime with fluxes. In contrast, the m -th string equation of the negative KdV hierarchy (6.30) has only one perturbative solution, $E_{\Gamma}^{(|m|)}$, for Λ non-zero. This has $z^{-2/(2|m|+1)}$ leading order behaviour and can be real-valued in *both* weak coupling regimes¹¹. When Λ is set to zero we find that this solution no longer exists, and instead we are forced choose one of the solutions associated with the lower members of the hierarchy. An interesting interpretation of this is that the lower solutions could correspond to different phases of the *same* theory.

In [78, 79] it was demonstrated that certain heterotic supercritical string theories are unstable and can suffer closed string tachyon condensation down to lower dimensional string theories. This is certainly analogous to what we may be seeing here: string theories of lower dimensionality existing as phases of higher dimensional theories. Perhaps when we set Λ to zero we are somehow forcing the theory to condense down to a lower dimensional theory? If so then it would be interesting to understand

¹⁰Thanks to Clifford Johnson for suggesting this interpretation.

¹¹This apparent symmetry between $u(z)$ and $u(-z)$ could be a sign that the theory has no Ramond ground state, such as in the (p, q) odd superminimal models for example [15].

the physical interpretation of Λ , and to uncover a dynamical mechanism for how the tachyon condensation occurs. How this works exactly would depend on whether the $E_\Gamma^{(l^m)}$ are themselves non-perturbatively well-defined. If they were not, then the analogy would be exact. If they were, then the way to look at things would be that higher-dimensional theories have a choice of stable vacua, some of which correspond to stable vacua associated with lower dimensionalities. It is perhaps possible that Λ is some kind of spacetime field¹², so that it would divide spacetime into domains characterised by different vacua and dimensionalities (though this is hard to visualise). If this were a dynamical process then an observer in the spacetime would presumably see some sort of decay process.

However, we should again be careful, because the endpoint of this condensation process could well be $E_\Gamma^{(0)}$ ($=F_\Gamma^{(0)}$) corresponding to the $u(z) = (4\Gamma^2 - 1)/(4z^2)$ solution, which has a pole. Recall that this is naively the $(2, 0)$ model and hence has infinite central charge. The string theory interpretation of this is unclear, but it would seem to correspond to an infinite dimensional theory. From the point of view of two-dimensional worldsheet gravity it corresponds to taking a semiclassical limit in which the partition function is dominated by stationary points of the action [66, 67]. Perhaps this is a sign of a fundamental sickness in the models, possibly caused by the physical tachyon. Or perhaps if we better understood the dynamical mechanism by which the condensation process worked in these theories then we would be able to rule out the $m = -1$ to $m = 0$ transition on physical grounds, such as Zamolodchikov's theorem that the central charge of a unitary theory cannot increase in any decay process [80].

In [78, 79] it was speculated that the endpoint of the condensation process in the heterotic string would be the critical theory. This certainly does not seem to be the case in our models. In any case, even the critical Type 0 theories have closed string physical tachyons and so are themselves unstable. It is perhaps this that the $m = -1$ to $m = 0$ transition is encoding: the theory lowering its dimensionality to try to reach the critical theory, but not being able to do this because the critical theory is itself problematic. So the endpoint of the condensation process is instead some semi-trivial theory with no obvious physical interpretation. These ideas have recently been presented in a paper by Polyakov [81], in which he discusses possible dimensional reductions of Type 0 theories. In [82] the endpoint of tachyon condensation in critical

¹²Thanks to Douglas Smith for suggesting this interpretation.

Type 0A string theory was speculated to be a so-called “bubble of nothing” [83]. This may be related to Type IIA string theory via a conjectured duality [82, 84]. However, the above discussion is extremely speculative, and so whether this really is a manifestation of tachyon condensation remains to be seen.

Finally, whilst we have already pointed out that the $F_{\Gamma}^{(|m|)}$ have non-perturbative problems, we have also seen that they in some way represent m -soliton solutions interpolating between various basic $m = 0$ vacua. It would be interesting to see if this had any physical relevance. In the same vein, it would also be interesting to uncover more about the physical role of the \hat{t}_n .

Whilst there are still many unanswered questions, it is very encouraging that the results obtained from the negative KdV string equations are even remotely sensible. It remains to be seen whether they are just a curious mathematical coincidence or something more physical. If they do describe some aspect of supercritical string theory then it would be very exciting indeed. Certainly the way the theory can apparently change its dimensionality, possibly via some sort of tachyon condensation, is intriguing. In future work it would be insightful to carry out a similar analysis for the Type 0B models associated to the Zakharov-Shabat integrable hierarchy. It might be beneficial to understand the similarities and differences to the Type 0A case. Finally, it would be very interesting to see if we could analytically continue still further to study models with fractional values of m . Intuitively though, one feels that this would not be possible.

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Appendix A

Additional Information for Chapter 1

A.1 What Is String Theory?

A zero-dimensional object, or particle, will sweep out a one-dimensional path $x^\mu(\tau)$ in D -dimensional spacetime as it moves. Here, the x^μ are the spacetime coordinates¹ parametrised by the timelike parameter τ . By analogy, the basic one-dimensional object in string theory, the string, will sweep out a two-dimensional *worldsheet* in flat Minkowski spacetime, $X^\mu(\sigma, \tau)$;² parametrised in terms of the *worldsheet coordinates* $\sigma = 0, \dots, l$ and $\tau \in \mathbb{R}$ (see Figure A.1). For open strings, $X^\mu(0, \tau)$ and $X^\mu(l, \tau)$ define the two ends of the string in spacetime, whereas for closed strings σ is defined to be periodic with $X^\mu(0, \tau) = X^\mu(l, \tau)$.³ In the literature, the X^μ are referred to as the *target spacetime* coordinates, whereas τ and σ are the *worldsheet coordinates*. In this regard, string theory can be thought of as a quantum field theory, with fields X^μ , living on the two dimensional worldsheet. This interpretation will be returned to later.

The simplest Poincaré invariant (i.e. Lorentz invariance plus translational invariance) action is the *Nambu-Goto* action [1], which is proportional to the area of the

¹ x^0 being time and x^μ , $\mu = 1 \dots D - 1$ being the $D - 1$ space coordinates

² X^0 plays the role of time.

³Accordingly, open string worldsheets have boundaries, whereas closed string worldsheets do not.

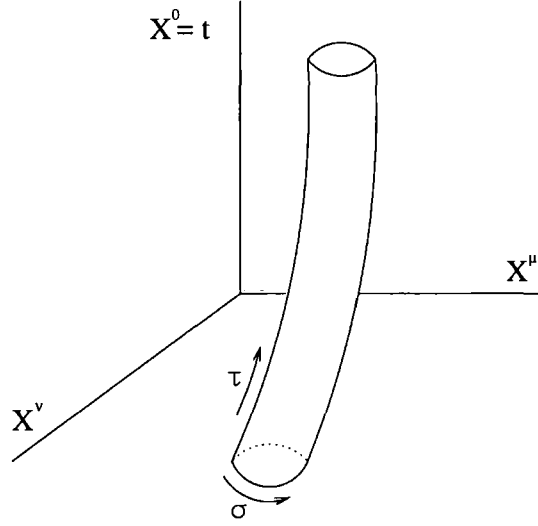


Figure A.1: A string worldsheet in target spacetime.

worldsheet:

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma (-\det h_{ab})^{1/2}, \quad (\text{A.1.1})$$

where the integration covers the whole worldsheet (open or closed), α' is a constant inversely proportional to the tension of the string, and h_{ab} is the *induced metric*, with Latin indices a and b running over (σ, τ) :

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu. \quad (\text{A.1.2})$$

Repeated indices are, as usual, to be summed over according to the Einstein convention. Classically, the functions X^μ will satisfy the usual equation of motion $\delta S / \delta X^\mu = 0$, which is the usual functional derivative. However, a problem with the Nambu-Goto action is its square root form. This will make quantisation difficult. To surmount this problem, let us consider the following *Polyakov* action:

$$S_{Pol} = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu, \quad (\text{A.1.3})$$

where we have introduced an auxiliary function $\gamma_{ab}(\tau, \sigma)$ known as the (*Lorentzian*) *worldsheet metric*, with $\gamma \equiv \det \gamma_{ab}$. We use γ_{ab} to raise and lower worldsheet indices. On the face of it, the actions of (A.1.1) and (A.1.3) look very different. However, using the equation of motion for γ_{ab} we have⁴:

$$\frac{\delta S_{Pol}}{\delta \gamma_{ab}} \equiv T_{ab} = 0 \quad \Rightarrow \quad h_{ab} = \frac{1}{2} \gamma_{ab} \gamma^{cd} h_{cd} \quad \Rightarrow \quad h_{ab} (-h)^{1/2} = \gamma_{ab} (-\gamma)^{1/2}. \quad (\text{A.1.4})$$

⁴Using $\delta \gamma = \gamma \gamma^{ab} \delta \gamma_{ab}$.

T_{ab} is known as the *energy-momentum tensor*. This can be used to eliminate γ_{ab} from the Polyakov action, which yields the result: $S_{Pol}[X, \gamma] = S_{NG}[X]$. That is, the Polyakov and Nambu-Goto actions are equivalent. The difference is that the Polyakov action lacks the troublesome square root that plagues the Nambu-Goto action. Before quantising the theory it is useful to consider invariances of the Polyakov action (A.1.3). Just like (A.1.1), the Polyakov action is Poincaré invariant. It also has what is known as worldsheet *diffeomorphism invariance*, which is invariance under reparameterisations, $\sigma' = \sigma'(\sigma, \tau)$, $\tau' = \tau'(\sigma, \tau)$, on the worldsheet:

$$\begin{aligned} X^\mu(\tau, \sigma) &\mapsto X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma), \\ \gamma_{ab}(\tau, \sigma) &\mapsto \gamma'_{ab}(\tau', \sigma'), \quad \text{with} \quad \frac{\partial \sigma'^c}{\partial \sigma^a} \frac{\partial \sigma'^d}{\partial \sigma^b} \gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma). \end{aligned} \quad (\text{A.1.5})$$

This invariance encodes the fact that the physics should be independent of the worldsheet coordinates. We also have *Weyl invariance*:

$$\begin{aligned} X^\mu(\tau, \sigma) &\mapsto X'^\mu(\tau, \sigma) = X^\mu(\tau, \sigma), \\ \gamma_{ab}(\tau, \sigma) &\mapsto \gamma'_{ab}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \gamma_{ab}(\tau, \sigma), \end{aligned} \quad (\text{A.1.6})$$

where $\omega(\tau, \sigma)$ is arbitrary. Using the diffeomorphism and Weyl invariances we have exactly enough freedom to fix the three independent components of the worldsheet metric γ_{ab} in the Polyakov action (A.1.3). The classical equations of motion associated with the Polyakov action are $T_{ab} = 0$ from variation of the action with respect to γ_{ab} , and, from variation with respect to X^μ :

$$\frac{\delta S_{Pol}}{\delta X^\mu} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma (-\gamma)^{1/2} \partial^a \partial_a X_\mu, \quad (\text{A.1.7})$$

which is valid in the case of closed strings. For open strings we also have a boundary term to add to $\delta S_{Pol}/\delta X^\mu$:

$$-\frac{1}{2\pi\alpha'} \int d\tau (-\gamma)^{1/2} \partial^\sigma X_\mu \Big|_{\sigma=0}^{\sigma=l}, \quad (\text{A.1.8})$$

For closed strings, and open strings away from their endpoints, (A.1.7) yields the equation of motion:

$$(-\gamma)^{1/2} \partial^a \partial_a X_\mu = 0, \quad (\text{A.1.9})$$

In the case of open strings we need the additional conditions:

$$\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, l) = 0, \quad (\text{A.1.10})$$

which are *Neumann* boundary conditions.

We could now proceed with a full analysis of the quantisation of the Polyakov action. However, this would be straying too far from the point of this introduction and so here we will merely summarise the salient points⁵. What will be most important in the context of this thesis will be the notion of Weyl invariance introduced above.

One can quantise the Polyakov action in a number of ways. In the simplest, known as *light-cone quantisation*, we define $X^\pm \equiv (X^0 \pm X^1)/\sqrt{2}$ and use up the diffeomorphism and Weyl freedom of the metric to choose:

$$X^+ = \tau, \quad \partial_\sigma \gamma_{\sigma\sigma} = 0, \quad \det \gamma_{ab} = -1, \quad (\text{A.1.11})$$

This is known as the *light-cone gauge*, and it has the advantage that we have fixed all the gauge freedom in the metric. The disadvantage is that we have lost the manifest covariance of the theory since we have picked out some of the target spacetime dimensions as in some way special. In this case the equations of motion in the other $D - 2$ dimensions, $X^i, i = 2 \dots D - 1$, reduce to free wave equations $\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i$, where $c = l/(2\pi\alpha'p^+)$. The constant p^+ here comes from the X^- equation of motion⁶. For open strings these equations of motion can be solved to give:

$$X^i(\tau, \sigma) = x^i + \frac{p_i \tau}{p^+} + i(2\alpha')^{1/2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \alpha_n^i e^{-\pi i n \sigma / l} \cos \frac{\pi n \sigma}{l}, \quad (\text{A.1.12})$$

where the x^i and p^i are constants (representing the overall centre-of-mass motion of the string), as are the $\alpha_n^i = (\alpha_{-n}^i)^\dagger$ (which relate to the vibration modes). Quantisation can be achieved by imposing equal time canonical commutation relations:

$$\begin{aligned} [x^-, p^+] &= -i, & [X^i(\sigma), \Pi^j(\sigma')] &= i\delta^{ij}\delta(\sigma - \sigma') \\ \Rightarrow [x^-, p^+] &= -i, & [x^i, p^j] &= \delta^{ij}, & [\alpha_m^i, \alpha_n^j] &= m\delta^{ij}\delta_{m, -n}, \end{aligned} \quad (\text{A.1.13})$$

where Π^j is the usual momentum conjugate to X_j . As in standard quantum field theory, the operators α_n^i have become creation ($n < 0$) and annihilation ($n > 0$) operators. The vacuum state of the theory is then $|0; k\rangle$, with centre-of-mass momenta

⁵For a far more detailed account refer to a textbook such as [1].

⁶It is actually the conjugate momentum associated to the X^- centre of mass, $x^- = \frac{1}{l} \int_0^l d\sigma X^-(\tau, \sigma)$.

$k = (k^+, k^i)$ (k^+ is like the energy of the string state) such that:

$$p^+|0; k\rangle = k^+|0; k\rangle, \quad p^i|0; k\rangle = k^i|0; k\rangle, \quad \alpha_n^i|0; k\rangle = 0, \quad m > 0. \quad (\text{A.1.14})$$

A general string state can be constructed from $|0; k\rangle$ by acting on it with the α_{-n}^i .

$$|N; k\rangle \propto \left[\prod_{i=2}^{D-1} \prod_{n=1}^{\infty} (\alpha_{-n}^i)^{N_{i,n}} \right] |0; k\rangle, \quad (\text{A.1.15})$$

where $N_{i,n}$ is the occupation number for the mode (i, n) , $i = 2, \dots, D-1$ and $n = 1, 2, \dots$. Each combination of the α_{-n}^i produces a different ‘particle’ in the target space. However, as we have already mentioned, quantisation by this method has obscured Lorentz covariance. It turns out that for the theory to still be Lorentz invariant after quantisation, we are forced to choose $D = 26$. That is, the dimension of the string theory itself must take on a specific value.

On analysis of the open string spectra we find that a string state has mass:

$$m^2 = \frac{1}{\alpha'} \left(N + \frac{2-D}{24} \right) \quad \text{with} \quad N = \sum_{i=2}^{D-1} \sum_{n=1}^{\infty} n N_{i,n}. \quad (\text{A.1.16})$$

N is known as the *level*. Using this equation, we see that the lightest string state, the scalar $|0; k\rangle$, has mass:

$$m^2 = \frac{2-D}{24\alpha'}. \quad (\text{A.1.17})$$

So for $D = 26$ the mass of the state is actually negative: it is a *tachyon*. This means that the vacuum of the theory is unstable. Note however that if $D \leq 2$ were somehow possible then this would not be the case. We will return to this later.

The simplest excited state is just $\alpha_{-1}^i|0; k\rangle$, which has zero mass. This is a good thing because it has exactly the $D-2$ components expected of a massless vector particle. If it were massive then we would expect $D-1$ components.

In the closed string case things are much the same, except that there are now two sets of operators β_n^i and $\tilde{\beta}_n^i$, corresponding to modes of vibration moving in opposite directions around the string. For obvious reasons, these are referred to as *left-movers* and *right-movers* respectively. A general state can be constructed in analogy to the open string sector:

$$|N, \tilde{N}; k\rangle \propto \left[\prod_{i=2}^{D-1} \prod_{n=1}^{\infty} (\beta_{-n}^i)^{N_{i,n}} (\tilde{\beta}_{-n}^i)^{\tilde{N}_{i,n}} \right] |0; k\rangle, \\ \text{with} \quad m^2 = \frac{2}{\alpha'} \left(N + \tilde{N} + \frac{2-D}{12} \right). \quad (\text{A.1.18})$$

This is supplemented by the *level-matching* condition $N = \tilde{N}$, which comes from imposing invariance under σ -translations on the states (which is true because any point on a closed string should be equivalent to any other). It turns out that the ground state of the theory is again tachyonic. The lightest excited state is $\beta_{-1}^i \beta_{-1}^j |0; k\rangle$, which is massless. It is a rank two tensor, and will actually turn out to be the *graviton* of the string theory.

The light-cone gauge quantisation explained above is the simplest method of quantising the string theory. Far more elegant is to use *path integral quantisation*, which proceeds in exactly the same way as in ordinary quantum field theory. We define the string partition function:

$$Z = \int \mathcal{D}X \mathcal{D}g e^{-S}, \quad (\text{A.1.19})$$

where we have performed functional integration over the ‘fields’ X and g .⁷ For more detailed discussion of this refer to any standard textbook (e.g. [85]). As usual, in the path integral formulation the classical equations of motion are not satisfied exactly. The classical solution is merely the most likely configuration of the system, and there will be fluctuations away from it. The partition function allows us to calculate expectation values of operators (the position X^μ for instance) simply by inserting them into the integrand in the partition function. This will be seen explicitly below where it will be shown that the path integral formulation is extremely powerful indeed.

Clearly it is possible to define the theory on worldsheets with various topologies. For instance, the worldsheet of our closed string could gain a handle (Fig. A.2(ℓ)). This would clearly correspond to the closed string propagating through spacetime before splitting into two closed strings which then rejoin to form a single closed string again. In the terminology of Feynman diagrams, this corresponds to a 1-loop diagram. Thus we see that string interactions are implicitly a part of string theory.

So far we have been considering a single string at $X^0 = -\infty$ propagating to $X^0 = \infty$. It is clear that by changing the topology of the worldsheet we can have any number of initial and final strings (Fig. A.2(r)). In the terminology of Feynman diagrams, these

⁷Here we have Wick rotated to a Euclidean worldsheet of signature $(+, +)$ with $\tau \mapsto i\sigma^1$, $\sigma \mapsto \sigma^2$. Consequently, g_{ab} is defined to be the Euclidean version of the Lorentzian worldsheet metric γ_{ab} used previously.

strings correspond to external legs. So far however, there is nothing in the theory controlling the strength of such processes. Indeed, it depends upon the worldsheet that we *choose* to define the string theory on. It turns out that the action of (A.1.3) is not the most general action to satisfy diffeomorphism and Weyl invariance. We can make the following generalisation:

$$\begin{aligned}
 Z &= \int \mathcal{D}X \mathcal{D}g e^{-S}, & S &= S_{Pol} + S_\chi, \\
 S_{Pol} &= \frac{1}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} g^{ab} \partial_a X^\mu \partial_b X_\mu, \\
 S_\chi &= \frac{\lambda}{4\pi} \int_M d^2\sigma g^{1/2} R + \frac{\lambda}{2\pi} \int_{\partial M} ds k,
 \end{aligned} \tag{A.1.20}$$

where λ is a constant, M denotes the worldsheet, and ∂M its boundary. Here, R is the two dimensional worldsheet Ricci scalar, ds is the proper time along the boundary, and k is the curvature of the boundary. This latter curvature term is of course only present in the case of open strings.

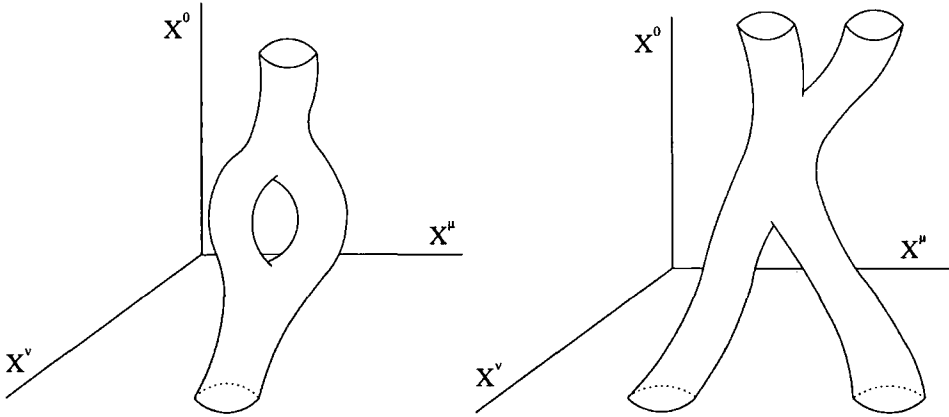


Figure A.2: (l) A closed string splits into two before rejoining; (r) two closed strings merge together before splitting apart again.

The fact that the worldsheet is two-dimensional is very important here, because it implies that the term involving R in the action is a total derivative. In fact, it is just λ multiplied the *Euler number*, χ , of the worldsheet: it is a *topological invariant*. For diagrams with no external legs χ is evaluated via the formula:

$$\chi = 2 - 2h - b, \tag{A.1.21}$$

where h is the number of handles on the worldsheet and b is the number of boundaries. We now see that e^λ is a coupling that controls the strength of the splitting and joining

of worldsheets. To see this, notice that adding a handle to the worldsheet weights the partition function by $e^{2\lambda}$. As was mentioned above, this is like a 1-loop Feynman diagram (Figure A.2a) encoding the emission and absorption of a virtual closed string.

Similarly, adding an extra boundary to the worldsheet is like the emission and absorption of a virtual open string. It weights the partition function by $e^\lambda \equiv g_s$. So the coupling associated with closed strings is the square of that associated with open strings. In the case of worldsheets with external legs, the Euler number is modified slightly, but we will not explore this here. Later we will see that g_s is not the free parameter that it first appears to be.

We therefore see that the proper definition of string theory should include topology changing processes. A more general definition of the partition function will therefore take into account a sum over every possible topology as well as the ‘sum’ over all possible metrics. Accordingly we alter our view of the partition function (1.9). Whereas before we thought of $\mathcal{D}g$ as integrating over all possible metrics on a given worldsheet, we will from now think of $\mathcal{D}g$ as implicitly integrating over all possible metrics on *all possible worldsheet topologies*. Specifically we only consider *vacuum diagrams*, which are worldsheets without external legs. Just like in ordinary quantum field theory, the diagrams with external legs, the n -point functions, can be derived once we know the vacuum partition function itself. In string theory, the insertion of initial and final states corresponds to the insertion of so-called *vertex operators* onto these vacuum worldsheets. This will be seen explicitly below. However, to really understand the full power of the partition function we will need to study some basic conformal field theory.

A.2 Conformal Field Theory and Vertex Operators in String Theory

It was mentioned above that the string theory can be thought of as a quantum field theory living on the string worldsheet and coupled to worldsheet gravity. In fact, this quantum field theory is an example of a *conformal field theory*, where the word *conformal* will be defined in the below. Therefore it is important to understand conformal field theory in order to understand string theory itself. In this section we

will meet some crucial topics such as *primary operators* and *central charge*. When we later study minimal string theories these aspects will prove to be very useful.

A conformal field theory (CFT) in two dimensions is defined by an action, $S[X^\mu]$, in terms of a set of fields $X^\mu(\sigma^1, \sigma^2)$, where σ^1 and σ^2 are Euclidean worldsheet coordinates with flat metric δ_{ab} . An important property of such a CFT is *conformal invariance*. To understand this, let us introduce a simple conformal field theory defined by the following action:

$$S_f = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu). \quad (\text{A.2.22})$$

In all two-dimensional CFTs it is useful to define complex coordinates $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$. We then have:

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2), \quad (\text{A.2.23})$$

conventionally one uses the simple notation $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$ providing this is unambiguous. The simple action (A.2.22) then becomes:

$$S_f = \frac{1}{2\pi\alpha'} \int d^2z \partial_1 X^\mu \bar{\partial} X_\mu, \quad (\text{A.2.24})$$

which yields the classical equation of motion:

$$\partial \bar{\partial} X^\mu = 0, \quad (\text{A.2.25})$$

which implies that ∂X^μ ($\bar{\partial} X^\mu$) is (anti)holomorphic. The action has a classical symmetry under worldsheet translations $\delta\sigma^a = \varepsilon v^a$, under which $\delta X^\mu = -\varepsilon v^a \partial_a X^\mu$, where the v^a are constants. Associated to this symmetry is the conserved Noether current:

$$j_a = i v^b T_{ab}, \quad T_{ab} = -\frac{1}{\alpha'} : \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \delta_{ab} \partial_c X^\mu \partial_c X_\mu \right) :. \quad (\text{A.2.26})$$

In this equation T_{ab} is known as the worldsheet energy-momentum tensor⁸. We have encountered it before in (A.1.4). Notice that it is traceless: $T_a^a = 0$, which can be re-written as $T_{z\bar{z}} = 0$. This tracelessness implies a much larger symmetry:

$$j(z) = i v(z) T(z), \quad \tilde{j}(\bar{z}) = i \bar{v}(z) \tilde{T}(\bar{z}), \quad (\text{A.2.27})$$

where $T(z) \equiv T_{zz}(z)$ is holomorphic, and $\tilde{T}(\bar{z}) \equiv \tilde{T}_{\bar{z}\bar{z}}(\bar{z})$ is antiholomorphic. Notice that in these conserved currents, $v(z)$ and $\bar{v}(z)$ are no longer constants. The symmetry

⁸The $::$ represent conformal normal ordering to be defined below.

associated with this conserved current is an infinitesimal coordinate transformation $z \mapsto z' = z + \varepsilon v(z)$. The finite transformation is:

$$X^\mu(z, \bar{z}) \mapsto X'^\mu(z', \bar{z}') = X^\mu(z, \bar{z}), \quad z' = f(z), \quad (\text{A.2.28})$$

for any holomorphic $f(z)$. Invariance under this transformation is known as *conformal invariance*. Physically, a conformal transformation stretches, contorts and rescales the worldsheet coordinates. Note that a mass term like $m^2 X^\mu X_\mu$ in the action would not be conformally invariant. One way of seeing this is that it comes with a dimensionful parameter m that would effectively scale under the transformation. In general, a theory will be conformally invariant if the stress-energy tensor is traceless.

Theories with this conformal invariance permit many simplifications that make them exactly solvable (in principle) in two dimensions. The most efficient way to quantise the theory is via the path integral approach. We calculate expectation values of quantities in a conformal field theory via the path integral:

$$\langle \mathcal{F}[X] \rangle = \int \mathcal{D}X e^{-S} \mathcal{F}[X]. \quad (\text{A.2.29})$$

Key ideas will be the concepts of *local operators* and the *operator product expansion*. Associated to any conformal field theory is a *complete set* of operators $\{\mathcal{A}_j(z, \bar{z})\}$, the expectation values of which are calculated via (A.2.29). The set of operators will include the identity operator and co-ordinate derivatives of each field involved in the action. By completeness we mean that any state of the theory can be generated by a linear action of these operators. Equivalently we have the operator algebra:

$$\mathcal{A}_i(z_1, \bar{z}_1) \mathcal{A}_j(z_2, \bar{z}_2) = \sum_k f_{ij}^k(z_1 - z_2, \bar{z}_1 - \bar{z}_2) \mathcal{A}_k(z_2, \bar{z}_2), \quad (\text{A.2.30})$$

where the $f_{ij}^k(z_1 - z_2, \bar{z}_1 - \bar{z}_2)$ are known as *coefficient functions*. This is useful when calculating correlation functions:

$$\langle \mathcal{A}_{i_1}(z_1, \bar{z}_1) \mathcal{A}_{i_2}(z_2, \bar{z}_2) \cdots \mathcal{A}_{i_n}(z_n, \bar{z}_n) \rangle. \quad (\text{A.2.31})$$

Ehrenfest's theorem states that upon quantisation the equations of motion hold as *operator equations*, so that the following expectation value result is true:

$$\langle \partial \bar{\partial} X^\mu(z, \bar{z}) \cdots \rangle = 0, \quad (\text{A.2.32})$$

where the \dots represent arbitrary operator insertions providing that said insertions are not coincident with $X^\mu(z, \bar{z})$. If this is the case then we find that (A.2.32) no longer holds. We can rectify this by introducing *conformal normal ordering*. Normal ordered operators are denoted $: \mathcal{A} :$. This is defined so that if the operators in (A.2.32) were normal ordered then they would satisfy the equation of motion even if there was some coincidence. For instance:

$$\begin{aligned} : X^\mu(z, \bar{z}) : &= X^\mu(z, \bar{z}), \\ : X^\mu(z_1, \bar{z}_1) X^\mu(z_2, \bar{z}_2) : &= X^\mu(z_1, \bar{z}_1) X^\mu(z_2, \bar{z}_2) + \frac{\alpha'}{2} \eta^{\mu\nu} \ln |z_1 - z_2|^2, \end{aligned} \quad (\text{A.2.33})$$

where $\eta^{\mu\nu}$ is the spacetime flat Minkowski metric.

Conformal invariance puts special constraints upon the form of the operator product expansion (A.2.31). Any product of operators with components of T_{ab} will be especially simple due to the (anti)holomorphy of $T(z)$ ($\tilde{T}(z)$), because the corresponding coefficient functions in (A.2.30) must also have this property. We can always choose a basis of operators $\{\mathcal{A}_j(z, \bar{z})\}$ that transform in a certain way under a rigid transformation, $z \mapsto z' = az$, for some complex parameter a . This transformation serves to rotate and rescale the coordinate system (but not deform it otherwise). We have:

$$\mathcal{A}_j(z, \bar{z}) \mapsto \mathcal{A}'_j(z', \bar{z}') = a^{-h_j} \bar{a}^{-\tilde{h}_j} \mathcal{A}_j(z, \bar{z}), \quad (\text{A.2.34})$$

where (h_j, \tilde{h}_j) are known as the *weights* of \mathcal{A}_j . The sum $h_j + \tilde{h}_j$ is known as the *dimension* of \mathcal{A}_j , determining its behaviour under scaling, and the difference $h_j - \tilde{h}_j$ is the *spin*, determining the behaviour under rotations. In what follows we will often suppress the \bar{z} argument of fields, which we can do without loss of generality because the same arguments apply for \bar{z} as apply for z . Under an infinitesimal general conformal transformation, $z \mapsto z' = z + \varepsilon v(z)$, the local properties ensure that the \mathcal{A}_j must transform as a linear combination of the function $v(z)$ and finite number of its derivatives [13]:

$$\delta \mathcal{A}_j(z) = \varepsilon \sum_{k=0}^{\nu_j} \mathcal{B}_j^{\{k-1\}}(z) \frac{d^k v(z)}{dz^k}, \quad (\text{A.2.35})$$

where the $\mathcal{B}_j^{\{k-1\}}$ are local fields belonging to the set $\{\mathcal{A}_j\}$, and ν_j is some integer. It is clear that the weights of the fields $\mathcal{B}_j^{\{k-1\}}$ are given by $h_j + 1 - k$ for $k = 0, 1, \dots, \nu_j$. A physically suitable theory should not possess correlations that increase with distance. This means that we must have $h_j \geq 0$ and $\tilde{h}_j \geq 0$ for every operator \mathcal{A}_j in our

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complete set. For this reason we see that $\nu_j \leq h_j + 1$. Notice also that, since the $\mathcal{B}_j^{\{k-1\}}$ also belong to the complete set, we see that the spectrum of weights, h_j , in any two-dimensional CFT consists of the infinite integer-spaced series:

$$h_j^{[k]} = \Delta_j + k, \quad k = 0, 1, 2, \dots, \quad (\text{A.2.36})$$

where Δ_j is the minimum weight in each series. So, we have towers of states all related to the operators with lowest weights, Δ_j . A special set of operators, \mathcal{O}_j , are known as *primary fields* or *tensor operators*. Under general conformal transformation they transform as:

$$\mathcal{O}_j(z, \bar{z}) \mapsto \mathcal{O}'_j(z', \bar{z}') = (\partial_z z')^{-h_j} (\partial_{\bar{z}} \bar{z}')^{-\bar{h}_j} \mathcal{O}_j(z, \bar{z}). \quad (\text{A.2.37})$$

Using (A.2.37) and (A.2.35) we see that under the infinitesimal conformal transformation a primary field satisfies:

$$\delta \mathcal{O}_j(z) = \varepsilon [v(z) \partial \mathcal{O}_j + h_j v'(z) \mathcal{O}_j], \quad (\text{A.2.38})$$

where any higher terms are absent, implying that this is an operator of lowest weight. So we now see that the operators forming the foundation of the tower of basic operators (A.2.36) are just the primary fields. In general, a theory will have some number of primary fields \mathcal{O}_j with weights $(\Delta_j, \tilde{\Delta}_j)$. Derived from these will be *conformal families* $[\mathcal{O}_j]$ of *secondary operators* with weights $(\Delta_j + k, \tilde{\Delta}_j + \tilde{k})$ for $k = 1, 2, \dots$. Under conformal transformations, each member of a conformal family transforms only in terms of members of the same family. There is no mixing between different families, so each family forms some irreducible representation of the conformal algebra. The sum of all the conformal families will constitute the complete set of operators \mathcal{A}_j .

For primary fields the following operator product is particularly simple:

$$T(z) \mathcal{O}_j(0, 0) = \sum_{k=0}^{\infty} z^{-2+k} \mathcal{O}_j^{\{-k\}}(0, 0) = \frac{\Delta_j}{z^2} \mathcal{O}_j(0, 0) + \frac{1}{z} \partial \mathcal{O}_j(0, 0) + \dots, \quad (\text{A.2.39})$$

with more singular terms being absent. Here we have determined the first two terms $\mathcal{O}_j^{\{0\}}$ and $\mathcal{O}_j^{\{-1\}}$ using (A.2.35). The higher $\mathcal{O}_j^{\{-k\}}$ coefficients will be new local fields in the same conformal family. Therefore they must be secondary operators: $\mathcal{O}_j^{\{-k\}} \in [\mathcal{O}_j]$, with weights $h_j^{\{k\}} = \Delta_j + k$. We can play the same game again by calculating the operator product $T(z) \mathcal{O}_j^{\{-k\}}(0, 0)$. This will be more complicated than before, but will contain new secondary fields $\mathcal{O}_j^{\{-k, -l\}}$. Using similar methods it is possible

to discover an infinite set of secondary fields of the form:

$$\mathcal{O}_j^{\{-k_1, -k_2, \dots, k_N\}}, \quad (\text{A.2.40})$$

where $k_i \geq 1$ and $N \geq 1$. These fields form the complete conformal family $[\mathcal{O}_j]$ and have weights $h_j^{k_1, k_2, \dots, k_N} = \Delta_j + k_1 + k_2 + \dots + k_N$. Not all the fields are linearly independent however. For instance:

$$\mathcal{O}_j^{\{-1, -k_1, k_2, \dots, k_N\}} = \partial \mathcal{O}_j^{\{-k_1, k_2, \dots, k_N\}}, \quad (\text{A.2.41})$$

so the conformal family naturally includes all the derivatives of each field involved.

To illustrate the above, let us write down the weights of some common operators:

$$\begin{aligned} X^\mu &\blacktriangleright (0, 0), & \partial X^\mu &\blacktriangleright (1, 0), \\ \bar{\partial} X^\mu &\blacktriangleright (0, 1), & :e^{ik \cdot X}: &\blacktriangleright \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right), \end{aligned} \quad (\text{A.2.42})$$

all of which also happen to transform as tensors⁹. Note also that the energy-momentum tensor, T_{ab} , is *not* actually a tensor in this respect. It has the name ‘tensor’ instead from its behaviour under the usual spacetime coordinate transformation. Let us also note that the only ‘good’ primary field with zero weights is the identity. Therefore, although we have assigned zero weights to the operator X^μ , it should be understood that it is technically not a ‘good’ primary field because it is not equivalent to the identity.

The final operator in (A.2.42), $:e^{ik \cdot X}:$, is what is known as a *vertex operator* in the string theory context. Insertion of it into the path integral (A.2.29) is equivalent to puncturing the worldsheet at its position (z, \bar{z}) . For instance, the basic closed string worldsheet, the infinite cylinder, is conformally equivalent to a twice-punctured sphere. We can think of these punctures as specifying the initial and final states of the string. So, if we were to insert the vertex operator $:e^{ik \cdot X}:$ onto the disc, we should interpret this as the preparation of an initial tachyon state with momentum k .

Different vertex operators have the form $:\mathcal{P}(\partial X, \partial^2 X, \dots; \bar{\partial} X, \bar{\partial}^2 X, \dots)e^{ik \cdot X}:$ and correspond to the insertions of various initial string states. For instance, the graviton of the closed string sector (A.1.18) has $:\partial X^\mu \bar{\partial} X^\nu e^{ik \cdot X}:$ as its vertex operator. Similarly, it is easy to see that initial and final open string states can be thought of as

⁹The operator $\partial^2 X^\mu$ transforming as $(2, 0)$ is an example of an operator that does not transform as a tensor.

the insertion of operators on worldsheet boundaries: so an infinite strip representing propagation of a single open string can be thought of as a disc with two boundary insertions. To include these open strings we of course need to add appropriate terms into the action, which are integrated over the boundary of the worldsheet, rather than its interior, such as the final term in (A.1.20). Correspondingly we can separate the action into a *bulk* theory and a *boundary* theory.

This mapping between states of the string theory and the basic operators is known as the *state-operator correspondence*. It can be shown to be bijective. Before using a vertex operator one should integrate over d^2z because it can be in any position on the worldsheet. This integral must be conformally invariant, and hence the vertex operator must have weight $(1, 1)$. For the tachyon vertex operator we therefore have (using (A.2.42)):

$$1 = \frac{\alpha' k^2}{4} \Rightarrow m^2 = -k^2 = -\frac{4}{\alpha'}, \quad (\text{A.2.43})$$

which tallies with the result obtained from the earlier light-cone gauge analysis. Using a similar method it is trivial to show that the closed string graviton is massless.

To examine the state-operator mapping in more detail we expand the $T(z)$ and $\tilde{T}(\bar{z})$ components of the energy-momentum tensor as:

$$T(z) = \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}}, \quad \tilde{T}(\bar{z}) = \sum_{m \in \mathbb{Z}} \frac{\tilde{L}_m}{\bar{z}^{m+2}}. \quad (\text{A.2.44})$$

The coefficients L_m and \tilde{L}_m are known as *Virasoro generators*. They satisfy the *Virasoro algebra*:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \\ [\tilde{L}_m, \tilde{L}_n] &= (m - n)\tilde{L}_{m+n} + \frac{\tilde{c}}{12}(m^3 - m)\delta_{m,-n}, \end{aligned} \quad (\text{A.2.45})$$

where the constants c and \tilde{c} are known as central charges. For most common theories $c = \tilde{c}$. The specific value of the central charge depends on the particular form of the conformal field theory action. In the case of free scalars X^μ , as in (A.2.22), the central charge is equal to D , the number of such fields (i.e. each scalar contributes unity to the central charge). Free fermions contribute a half to the central charge. If a field in

the action is not free then the central charge can take various values.

So we see from (A.2.27) that the L_m form an infinite set of conserved charges for the conformal field theory. We will see in Chapter 2 how this suggests that the theory is *integrable*, which would mean that the CFT is exactly solvable¹⁰. The Virasoro operators form an appropriate Hilbert space that turns out to be closely related to the space of string states. We can define a vacuum $|0\rangle$ state that is annihilated by L_m if $m \geq -1$. This must be the case else the stress-energy tensor would have been singular at $z = 0$. This is the manifestation of the conformal invariance of the vacuum. Using the state-operator mapping we see that to the primary fields we can associate *primary states*, also known as vectors:

$$|j\rangle = \mathcal{O}_j(0)|0\rangle. \quad (\text{A.2.46})$$

A primary state is also known as a *highest weight state* because it is annihilated by all lowering operators L_m with $m > 0$. It also satisfies $L_0|j\rangle = \Delta_j|j\rangle$. What is more, the secondary fields in the associated conformal family are given by:

$$\mathcal{O}_j^{\{-k_1, -k_2, \dots, k_N\}\{-\bar{k}_1, -\bar{k}_2, \dots, \bar{k}_N\}}(0)|0\rangle = L_{-k_1} L_{-k_2} \cdots L_{-k_N} \tilde{L}_{-\bar{k}_1} \tilde{L}_{-\bar{k}_2} \cdots \tilde{L}_{-\bar{k}_N} |n\rangle, \quad (\text{A.2.47})$$

where we have restored the operators associated with \bar{z} that we had suppressed above. A key result is that one can express any correlation function involving any combination of operators in $\{\mathcal{A}_j\}$ in terms of the correlators of primary fields \mathcal{O}_j . For instance:

$$\left\langle \mathcal{O}_j^{\{-k_1, \dots, k_M\}}(z) \prod_{i=1}^N \mathcal{O}_i(z_i) \right\rangle = \hat{\mathcal{L}}_{-k_M} \cdots \hat{\mathcal{L}}_{-k_1} \left\langle \mathcal{O}_j(z) \prod_{i=1}^N \mathcal{O}_i(z_i) \right\rangle, \\ \text{with } \hat{\mathcal{L}}_{-k}(z, z_i) = \sum_{i=1}^N \left[\frac{(1-k)\Delta_i}{(z-z_i)^k} - \frac{1}{(z-z_i)^{k-1}} \frac{\partial}{\partial z_i} \right]. \quad (\text{A.2.48})$$

So all information about the CFT is contained in the correlators of primary fields.

We illustrate the above by studying the specifics of (A.2.22). We can expand the holomorphic ∂X^μ and the antiholomorphic $\bar{\partial} X^\mu$ as:

$$\partial X^\mu(z) = -i \left(\frac{\alpha'}{2} \right)^{1/2} \sum_{m \in \mathbb{Z}} \frac{\beta_m^\mu}{z^{m+1}}, \quad \bar{\partial} X^\mu \bar{z} = -i \left(\frac{\alpha'}{2} \right)^{1/2} \sum_{m \in \mathbb{Z}} \frac{\tilde{\beta}_m^\mu}{\bar{z}^{m+1}}, \quad (\text{A.2.49})$$

¹⁰Although this would not necessarily imply that the string theory itself is exactly solvable, because one also has to take into account the sum over worldsheets.

where we have assumed closed string boundary conditions. We can integrate to obtain:

$$X^\mu(z) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{1}{m} \left(\frac{\beta_m^\mu}{z^{m+1}} + \frac{\tilde{\beta}_m^\mu}{\bar{z}^{m+1}} \right). \quad (\text{A.2.50})$$

The operator product $X^\mu X^\nu$ then yields the commutation rules:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\beta_m^\mu, \beta_n^\nu] = [\tilde{\beta}_m^\mu, \tilde{\beta}_n^\nu] = m\eta^{\mu\nu} \delta_{m,-n} \quad (\text{A.2.51})$$

In the case of open string boundary conditions one has the additional constraint $\alpha_m^\mu = \tilde{\beta}_m^\mu$. The L_m are given in terms of sums of products of the β_n^μ :

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \beta_{m-n}^\mu \beta_{\mu n} :, \quad (\text{A.2.52})$$

and similarly for \tilde{L}_m . Here the normal ordering refers to the usual creation-annihilation normal ordering of quantum field theory: placing lowering operators to the right of raising operators. The two forms of normal ordering turn out to be equivalent in the relatively simple CFT studied here. Notice the similarity in the above to the formalism of (A.1.14). In this section we have seen schematically that it is possible to derive the string states, found earlier via light-cone quantisation, by instead using the properties of operator product expansions.

Finally in this section, let it be stated that a conformal field theory with a positive definite inner product between states in its Hilbert space is known as a *unitary CFT*. These CFTs must have central charges that are greater than or equal to zero.

A.3 Complications: Moduli and CKVs

In this section we will review some further complications that arise when quantising String theory. These complications stem from the fact that, on certain surfaces, the space of possible metrics and the space of diffeomorphism and Weyl transformations are not quite the same. This mismatch can occur in two possible ways: either not all the metrics are equivalent under diffeomorphism and Weyl transformations; or certain diffeomorphism and Weyl transformations will not change the metric at all.

Let us consider a torus defined over the coordinate region $0 \leq \sigma_1 \leq 2\pi$ and $0 \leq \sigma_2 \leq 2\pi$ with periodic boundary conditions. We can define this as the σ -plane with the

identification $(\sigma^1, \sigma^2) \equiv (\sigma^1 + 2\pi m, \sigma^2 + 2\pi n)$. The key point is that we can equally well define the torus as the σ -plane with the identification:

$$(\sigma^1, \sigma^2) \equiv (\sigma^1 + 2\pi m, \sigma^2 + 2\pi \tau n), \quad (\text{A.3.53})$$

for any complex *modulus* τ . Each value of τ actually defines a distinct metric. However, the diffeomorphism and Weyl transformations do not change the value of τ ; so, in order for the partition function to be complete, we need to integrate over τ . In general a surface will have several of these moduli that cannot be removed by the symmetries. Actually, not all moduli are inequivalent. In the case of the torus it turns out that τ and τ' are equivalent where:

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1. \quad (\text{A.3.54})$$

That is, the torus modulus is invariant under a $SL(2, \mathbb{Z})$ transformation. That this is true can be seen by replacing $(m, n) \mapsto (m + n, n)$ and $(m, n) \mapsto (m, -n)$ in (A.3.53). There are also symmetries of the worldsheet that are not fixed by the choice of metric. An example of this is that a torus worldsheet metric is left invariant by *rigid* translations of the form $\sigma^a \mapsto \sigma^a + v^a$, where v^a is a constant. Similarly, the metric is invariant under $\sigma^a \mapsto -\sigma^a$. These unfixed symmetries are known as the *Conformal Killing Group*, and the infinitesimal forms of each transformation are known as *Conformal Killing Vectors* (CKVs): we cannot fix them by our choice of metric. The torus therefore has two CKVs. We can account for the Conformal Killing Group in calculations by explicitly fixing some of the vertex operators on the surface (one for each residual symmetry). So the more handles and boundaries on a surface the more vertex operators it contains and hence the fewer residual symmetries it will have.

The *Riemann-Roch* theorem states that the number of CKVs, κ , the number of real moduli μ (a complex moduli counts as two real moduli), and the Euler number are related by the simple formula:

$$\kappa - \mu = 3\chi. \quad (\text{A.3.55})$$

What is more, κ vanishes for $\chi < 0$; and μ vanishes for $\chi > 0$ (e.g. the sphere and disc have no moduli).

Appendix B

Additional Information for Chapter 2

B.1 Noncanonical Coordinates and Symplectic Manifolds

For certain constrained problems it proves useful to work in noncanonical coordinates that have a more complex Poisson bracket structure. To do this we follow the procedure outlined in [12] and define a new set of $2n$ coordinates y^μ such that $y_i \equiv q_i$, $y_{i+n} \equiv p_i$ for $i = 1, 2, \dots, n$. The relations of subsection 2.1.1 give:

$$\{y^\mu, y^\nu\} = \epsilon^{\mu\nu}, \quad \text{with} \quad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (\text{B.1.1})$$

where I is the $N \times N$ identity matrix. The chain rule then implies:

$$\{F(y), G(y)\} = \partial_\mu F \{y^\mu, y^\nu\} \partial_\nu G = \epsilon^{\mu\nu} \partial_\mu F \partial_\nu G, \quad (\text{B.1.2})$$

where $\partial_\mu \equiv \partial/\partial y^\mu$ in the usual way.

Choosing a noncanonical set of coordinates amounts to making the antisymmetric $\epsilon_{\mu\nu}$ tensor dependent on the y^μ 's. Indeed, this serves as a definition of noncanonical coordinates. In this case we can write the *noncanonical* Poisson bracket as:

$$\{F(y), G(y)\} = \partial_\mu F \{y^\mu, y^\nu\} \partial_\nu G = f^{\mu\nu} \partial_\mu F \partial_\nu G \quad (\text{B.1.3})$$

The antisymmetry property of Poisson brackets implies that $f^{\mu\nu} = -f^{\nu\mu}$, and the linear independence of the y^μ implies that $f_{\mu\nu}$ must be nonsingular, that is $f_{\mu\lambda} f^{\lambda\nu} =$

δ_μ^ν , where $f_{\mu\nu}$ is the inverse matrix. Finally, we require that the Jacobi identity be satisfied:

$$\{y^\mu, \{y^\nu, y^\lambda\}\} + \{y^\lambda, \{y^\mu, y^\nu\}\} + \{y^\nu, \{y^\lambda, y^\mu\}\} = 0, \quad (\text{B.1.4})$$

implying:

$$\partial_\mu f_{\nu\lambda} + \partial_\lambda f_{\mu\nu} + \partial_\nu f_{\lambda\mu} = 0, \quad (\text{B.1.5})$$

which is the Bianchi identity. Note that $f^{\mu\nu}$ can be thought of as a metric tensor, albeit an anti-symmetric one. Like a metric, we can use it to raise and lower indices. This defines what is known as a *symplectic manifold*.

B.2 Integrability of The KdV Equation

To prove that the KdV equation is indeed an integrable system we need to find an infinite number of conserved quantities. In the context of KdV, a conserved quantity will be a functional $\mathcal{H}[u]$ satisfying the relation:

$$\frac{d\mathcal{H}[u]}{dt} = \{\mathcal{H}[u], H\} = 0. \quad (\text{B.2.6})$$

Thinking of $\mathcal{H}[u]$ as a conserved charge, we can write it as the integral over a charge density, $\rho[u(x, t)]$:

$$\mathcal{H}[u] = \int_{-\infty}^{\infty} dx \rho[u(x, t)], \quad (\text{B.2.7})$$

which satisfies the standard continuity equation of electrodynamics:

$$\frac{\partial \rho[u(x, t)]}{\partial t} + \frac{\partial j[u(x, t)]}{\partial x} = 0, \quad (\text{B.2.8})$$

where $j[u(x, t)]$ is the analogue of a ‘current density’.

So, if we can find equations of the form (B.2.8) then we can find a conserved quantity. The most obvious example of this comes from the KdV equation (2.11) itself. It can be simply re-written as:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(3u^2 - \frac{\partial^2 u}{\partial x^2} \right). \quad (\text{B.2.9})$$

So we can identify:

$$\rho_0 = u(x, t), \quad j_0 = \frac{\partial^2 u}{\partial x^2} - 3u^2, \quad \mathcal{H}_0[u] = \int_{-\infty}^{\infty} dx u(x, t). \quad (\text{B.2.10})$$

Therefore $\mathcal{H}_0[u]$ is our first constant of motion. A second follows quickly by multiplying (2.11) by $u(x, t)$ and integrating:

$$\rho_1 = -\frac{1}{2}u(x, t)^2, \quad j_1 = \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + 2u^3 - u \frac{\partial^2 u}{\partial x^2}, \quad (\text{B.2.11})$$

$$\mathcal{H}_1[u] = -\frac{1}{2} \int_{-\infty}^{\infty} dx u(x, t)^2. \quad (\text{B.2.12})$$

Notice that $\mathcal{H}_1[u]$ is just the alternative Hamiltonian, \hat{H} , of (2.16). A third constant of motion is merely the Hamiltonian, $H[u]$, defined in (2.14). This is because of the trivial relation:

$$\frac{dH}{dt} = \{H, H\} = 0. \quad (\text{B.2.13})$$

After the first three integrals of motion, matters get much more complicated. Eleven were originally constructed in the above manner before a more systematic approach was developed to find the others. One method involves the study of the so-called *Miura map* [12], which relates solutions of the KdV equation to those of the associated *modified* KdV equation, or mKdV for short. The Miura map and the mKdV equation are discussed in subsection 2.3.2 and will prove to be of vital importance in the string theory context. In (2.4.1) we will derive the conserved quantities in a far more elegant and powerful way.

However, it turns out that we can prove the integrability of the KdV equation without knowing all the integrals of motion explicitly. We note that all of the three conserved quantities above, \mathcal{H}_0 , $\mathcal{H}_1 \equiv \hat{H}$ and $\mathcal{H}_2 \equiv H$, satisfy the following functional recursion relation:

$$(\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_{i-1}}{\delta u(x)} = \partial \frac{\delta \mathcal{H}_i}{\delta u(x)} \quad i = 0, 1, 2, \quad (\text{B.2.14})$$

where we have defined $\partial \equiv \partial/\partial x$ and $u_x \equiv \partial u/\partial x$ for clarity¹. We have also set $\mathcal{H}_{-1} = 0$. Given (B.2.14), it makes sense to ask whether or not any of the other conserved quantities obey the same recursion relation. This is indeed the case, and it can be proved by a simple inductive argument. We assume that integrals of motion,

¹Later on when we differentiate quantities with subscripts, such as S_i for example, we will write its differential as $S_{i,x}$ to avoid confusion.

\mathcal{H}_i , are known for $i = 0, 1, \dots, n$, and that they satisfy (B.2.14). \mathcal{H}_{n-1} and \mathcal{H}_n then satisfy:

$$(\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_{n-1}}{\delta u(x)} = \partial \frac{\delta \mathcal{H}_n}{\delta u(x)}. \quad (\text{B.2.15})$$

Since \mathcal{H}_n is known to be an integral of motion we know:

$$\frac{d\mathcal{H}_n}{dt} = \{\mathcal{H}_n, H\} = [\mathcal{H}_n, \hat{H}] = 0, \quad (\text{B.2.16})$$

which gives:

$$\frac{d\mathcal{H}_n}{dt} = [\mathcal{H}_n, \hat{H}] = - \int_{-\infty}^{\infty} dx \frac{\delta \hat{H}}{\delta u(x)} (\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_n}{\delta u(x)}, \quad (\text{B.2.17})$$

$$= \int_{-\infty}^{\infty} dx u (\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_n}{\delta u(x)}. \quad (\text{B.2.18})$$

In order to vanish, the integrand in (B.2.18) must be a total derivative. The only way we can achieve this is by writing:

$$(\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_n}{\delta u(x)} = \frac{\delta \mathcal{R}_n}{\delta u(x)}, \quad (\text{B.2.19})$$

$$\text{with } \mathcal{R}_n[u] = \int_{-\infty}^{\infty} dx \mathbb{L}(u), \quad (\text{B.2.20})$$

where $\mathbb{L}(u)$ is a function of u and its derivatives. That this works can be seen by substituting (B.2.19) into (B.2.18), integrating by parts and then using:

$$\frac{d\mathbb{L}}{dx} = u_x \frac{\partial \mathbb{L}}{\partial u} + u_{xx} \frac{\partial \mathbb{L}}{\partial u_x} + \dots, \quad (\text{B.2.21})$$

$$= u_x \frac{\partial \mathbb{L}}{\partial u} + u_x \frac{\partial}{\partial x} \frac{\partial \mathbb{L}}{\partial u_x} + \dots = u_x \frac{\delta \mathcal{R}_n}{\delta u(x)}, \quad (\text{B.2.22})$$

where the second line is only equal to the first up to total derivatives. Note the similarity between (B.2.19) and (B.2.14). It is tempting to enquire whether \mathcal{R}_n is a conserved quantity too. To do this we need to evaluate $d\mathcal{R}_n/dt = \{\mathcal{R}_n, H\}$. Let us consider the following manipulation:

$$\begin{aligned} \{\mathcal{H}_m, \mathcal{H}_n\} &= \int_{-\infty}^{\infty} dx \frac{\delta \mathcal{H}_m}{\delta u(x)} \partial \frac{\delta \mathcal{H}_n}{\delta u(x)}, \\ &= \int_{-\infty}^{\infty} dx \frac{\delta \mathcal{H}_m}{\delta u(x)} (\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_{n-1}}{\delta u(x)}, \\ &= - \int_{-\infty}^{\infty} dx \frac{\delta \mathcal{H}_{n-1}}{\delta u(x)} (\partial^3 - 4u\partial - 2u_x) \frac{\delta \mathcal{H}_m}{\delta u(x)}, \\ &= - \int_{-\infty}^{\infty} dx \frac{\delta \mathcal{H}_{n-1}}{\delta u(x)} \partial \frac{\delta \mathcal{H}_{m+1}}{\delta u(x)} = \{\mathcal{H}_{m+1}, \mathcal{H}_{n-1}\}, \end{aligned} \quad (\text{B.2.23})$$

which is a relationship we can use to show that $\{\mathcal{H}_m, \mathcal{H}_n\} = \{\mathcal{H}_n, \mathcal{H}_m\} = 0$. The same method allows us to prove that $[\mathcal{H}_m, \mathcal{H}_n] = [\mathcal{H}_n, \mathcal{H}_m] = 0$ also. This shows that the conserved quantities are in involution, which was one of the requirements for integrability. We can also use the same manipulation to show that $\{\mathcal{R}_n, H\} = \{H, \mathcal{R}_n\} = 0$, which completes the inductive proof that if \mathcal{H}_m is an integral of motion then there always exists another integral of motion, \mathcal{R}_n , defined by (B.2.19), that is also an integral of motion. It remains to show that \mathcal{R}_n is an independent conserved quantity. We can do this by considering its *scaling dimension* (c.f. Section 2.2). Using (B.2.14) we obtain:

$$\left[\frac{\delta \mathcal{H}_n}{\delta u(x)} \right] = \left[\frac{\delta \mathcal{H}_{n-1}}{\delta u(x)} \right] - 2[x] = \left[\frac{\delta \mathcal{H}_0}{\delta u(x)} \right] - 2n[x] = [1] - 2n[x] = -2n[x]. \quad (\text{B.2.24})$$

Since we know that $[\mathcal{R}_n] = [\mathcal{H}_n] - 2[x]$ we can eliminate the possibility that \mathcal{R}_n is dependent on any of the previously known \mathcal{H}_i because they cannot possibly scale in the correct manner. This means that we can safely assign $\mathcal{R}_n \equiv \mathcal{H}_{n+1}$, and by repeated application of (B.2.14) we can in the same way generate an infinite number of independent conserved quantities that are in involution. This completes the inductive argument and proves that the KdV equation meets all the necessary requirements to be completely integrable.

B.3 Matrix Operators and The Zakharov-Shabat Hierarchy

It is possible to consider a far more general set of hierarchies by extending the operator of (2.40) to include matrix-valued coefficients:

$$Q_n^{[m]}(-i\partial) = \sum_{k=0}^n U_k(x)(-i\partial)^k, \quad (\text{B.3.25})$$

where the U_i are $m \times m$ matrix-valued functions for some m . Operators of this type were first studied extensively by Gelfand and Dickey [86]. The method of [86] is highly elegant and involves the use of *symbols*, which are mathematical objects possessing special properties that we will not discuss here. The basic idea though is to replace $Q_n^{[m]}(\partial)$ in (B.3.25) by its abstract symbol equivalent $Q_n^{[m]}(\xi)$. This is done on the grounds that the definition of an operator such as $Q_n^{[m]}$ in (B.3.25) is non-local and requires boundary conditions.

The next step is to search for the operator $b(x, \xi, z) \equiv [Q_n^{[m]}(\xi) - z^n]^{-1}$, and its integrals:

$$S_k(x, z) = \int d\xi \xi^k b(x, \xi, z), \quad (\text{B.3.26})$$

where $S_0(x, z) \equiv R(x, z)$ is known as the *resolvent*, which is a little like the Green's function of the operator Q_n . One can expand the S_k as power series in z :

$$S_k(x, z) = \sum_{p=-k}^{\infty} S_{k,p} z^{-p-n+1}, \quad (\text{B.3.27})$$

and from here some calculation allows one to define the system of integrable equations (*flow equations*) corresponding to the operator defined in (B.3.25):

$$\begin{aligned} \frac{\partial}{\partial t_p} U_k = \sum_{r=k+1}^n \sum_{\alpha=0}^{r-k-1} \left[i^\alpha \binom{r-k-1}{\alpha} S_{r-k-\alpha-1, p-n} U_r^{(\alpha)} \right. \\ \left. - (-i)^\alpha \binom{r}{\alpha} U_r S_{r-k-\alpha-1, p-n}^{(\alpha)} \right] \frac{p-1}{n}, \end{aligned} \quad (\text{B.3.28})$$

where there is a flow equation for each value of p in the time variables t_p . We can obtain the S_k themselves from the following recursive formulae:

$$\begin{aligned} \sum_{k=0}^n \sum_{\alpha=0}^{\infty} (-i)^\alpha \binom{k}{\alpha} U_k S_{k+l-\alpha}^{(\alpha)} = z^n S_l, \\ \sum_{k=0}^n \sum_{\alpha=0}^{\infty} i^\alpha \binom{k+l}{\alpha} S_{k+l-\alpha} U_k^{(\alpha)} = z^n S_l, \end{aligned} \quad (\text{B.3.29})$$

where $l = 0, \pm 1, \pm 2, \dots$

Finally, the conserved quantities of the system can be written as:

$$\mathcal{H}_p = \int \text{Tr} R_p(x, z) dx. \quad (\text{B.3.30})$$

As is to be expected, the case $m = 1$ yields the KdV-type hierarchies defined via the pseudodifferential operator formalism of subsection 2.4.1. Now, however, we have power to define many more hierarchies. For instance, let us consider the $n = 1$ case:

$$Q_1^{[m]} = U_1 \partial - U_0, \quad (\text{B.3.31})$$

where U_1 is a constant matrix. Substitution into (B.3.29) yields the following two equations:

$$(U_0(x) - zI)S_0 + U_1 S_1 - iU_1 S_{0,x} = 0, \quad S_1 U_1 + S_0(U_0(x) - zI) = 0, \quad (\text{B.3.32})$$

where I is the $m \times m$ identity matrix. Eliminating S_1 from these gives:

$$(U_0(x) - zI)RU_1 - iU_1R_x - U_1R(U_0(x) - zI) = 0, \quad (\text{B.3.33})$$

and furthermore:

$$-i(U_1R)_x = (U_1R)(U_0(x) - zI)U_1^{-1} - (U_0(x) - zI)U_1^{-1}(U_1R). \quad (\text{B.3.34})$$

Finally we obtain the result:

$$-i(U_1R)_x = [U_1R, (U_0(x) - zI)U_1^{-1}]. \quad (\text{B.3.35})$$

Choosing U_1 to be diagonal and with $U_1^{-1} \equiv A = \text{diag}(a_1, a_2, \dots, a_m)$, (B.3.35) can be written in the following more compact form:

$$-\hat{R}_x + [U + zA, \hat{R}] = 0 \quad \Rightarrow \quad [\hat{Q}_1^{[m]}, \hat{R}] = 0, \quad (\text{B.3.36})$$

where $\hat{Q}_1^{[m]} \equiv -(Q_1^{[m]} - z)A = -\partial + U(x) + zA$, $\hat{R} = U_1R$, and $U(x) \equiv -U_0(x)A$. Using (B.3.27) we can re-write (B.3.36) as the following recursion relation:

$$-\hat{R}_{i,x} + [U, \hat{R}_i] = [\hat{R}_{i+1}, A], \quad i = -1, 0, 1, 2, \dots \quad (\text{B.3.37})$$

Since \hat{R}_{-1} is zero, the first of these equations, (B.3.37), for $i = -1$, implies that \hat{R}_0 is diagonal. Writing it in the form $\hat{R}_0 = \text{diag}(b_1, b_2, \dots, b_n)$ we can proceed to solve the $i = 0$ equation, first for its diagonal part (which tells us that the b_i are constants), and then for its non-diagonal part. Proceeding in this way, solving alternately for the diagonal and non-diagonal components of \hat{R} at each value of i , we find that the diagonal components of the \hat{R}_i are only ever determined up to addition of constants. These we can set to zero however, since their only effect is to mix any given \hat{R}_i with lower \hat{R}_i . The first few are [10]:

$$\begin{aligned} (\hat{R}_0)_{jk} &= b_j \delta_{jk}, & (\hat{R}_1)_{jk} &= \frac{b_j - b_k}{a_j - a_k} U_{jk} \quad (j \neq k), & (\hat{R}_1)_{jj} &= 0, \\ (\hat{R}_2)_{jk} &= -\frac{b_j - b_k}{(a_j - a_k)^2} U'_{jk} - \sum_{\beta \neq j, k} \frac{U_{j\beta} U_{\beta k}}{a_j - a_k} \left(\frac{b_j - b_\beta}{a_j - a_\beta} - \frac{b_\beta - b_k}{a_\beta - a_k} \right) \quad (j \neq k), \\ (\hat{R}_2)_{jj} &= \sum_{\beta \neq j} \frac{b_\beta - b_j}{(a_\beta - a_j)^2} U_{j\beta} U_{\beta j}. \end{aligned} \quad (\text{B.3.38})$$

The flow equations (B.3.28) for this system can be written as:

$$\partial_{t_p} U = [A, \hat{R}_{p-1}], \quad (\text{B.3.39})$$

where we have absorbed the factor of $(p - 1)$ present in (B.3.28) into the t_p .²

Let us now reduce the above first-order hierarchy (B.3.31) to a matrix sub-manifold, namely, a Lie subalgebra. We will study the case where the resolvent \hat{R} belongs to $sl(n, \mathbb{C}) \equiv su(n, \mathbb{C})$. Here the generators are the usual Pauli matrices with the commutation rules $[\sigma_i, \sigma_j] = \epsilon_{ij} \sigma_k$ and σ_1 being diagonal. We choose $U = f\sigma_2 + g\sigma_3$ and $\hat{R}_p = H_p\sigma_1 + F_p\sigma_2 + G_p\sigma_3$. The recursion relation (B.3.37) takes the form:

$$-H_{p,x} + fG_p - gF_p = 0, \quad (\text{B.3.40})$$

$$-F_{p,x} + gH_p = G_{p+1}, \quad (\text{B.3.41})$$

$$-G_{p,x} - fH_p = -F_{p+1}, \quad (\text{B.3.42})$$

the first few solutions of which are:

$$\begin{aligned} F_0 &= 0, & G_0 &= 0, & H_0 &= 1, \\ F_1 &= f, & G_1 &= g, & H_1 &= 0, \\ F_2 &= g_x, & G_2 &= -f_x, & H_2 &= -\frac{1}{2}(f^2 + g^2), \\ F_3 &= -f_{xx} - \frac{1}{2}f(f^2 + g^2), & G_3 &= -g_{xx} - \frac{1}{2}g(f^2 + g^2), & H_3 &= gf_x - fg_x, \\ F_4 &= -g_{xxx} - \frac{3}{2}g_x(f^2 + g^2), & G_4 &= f_{xxx} + \frac{3}{2}f_x(f^2 + g^2) \\ H_4 &= ff_{xx} + gg_{xx} - \frac{1}{2}f_x^2 - \frac{1}{2}g_x^2 + \frac{3}{8}(f^2 + g^2)^2. \end{aligned} \quad (\text{B.3.43})$$

The flow equations (B.3.39) can then be written as:

$$\begin{aligned} f_{t_{k+1}} - F_{k,x} + gH_k &= 0, \\ g_{t_{k+1}} - G_{k,x} - fH_k &= 0. \end{aligned} \quad (\text{B.3.44})$$

This hierarchy of equations is known as the *Zakharov-Shabat* (ZS) hierarchy. We will see it crop up again in the context of minimal string theory. The simplest non-trivial member of the hierarchy is the $k=2$ case:

$$\begin{aligned} f_{t_3} - g_{xx} - \frac{1}{2}g(f^2 + g^2) &= 0, \\ g_{t_3} + f_{xx} + \frac{1}{2}f(f^2 + g^2) &= 0. \end{aligned} \quad (\text{B.3.45})$$

²Which is possible because we have no interest in the singular case $p = 1$ since it is trivial.

This can be rewritten by letting $q = f + ig$:

$$-iu_{t_3} + u_{xx} - \frac{1}{2}u|u|^2 = 0, \quad (\text{B.3.46})$$

which is the celebrated *non-linear Schrödinger equation*.

The $k=3$ member of the ZS hierarchy is:

$$u_{t_4} + u_{xxx} + \frac{3}{2}u_x|u|^2 = 0. \quad (\text{B.3.47})$$

Upon restriction to real u ($g = 0$) we obtain:

$$f_{t_4} + f_{xxx} + \frac{3}{2}f_x f^2 = 0, \quad (\text{B.3.48})$$

which is just the (rescaled) mKdV equation of (2.29).

B.4 The Nature of The Spectral Parameter

For solitary wave solutions of KdV in the form $u(x, t) = f(x + ct)$ it can be shown [12] that *all* of the conserved quantities are proportional to the height of the wave, f , itself. We will use this fact below. Firstly, let us construct a one-parameter family of solutions, $u^{[\epsilon]}(x, t)$, to the KdV equation (2.11):

$$-4u_t = 6uu_x - u_{xxx}, \quad (\text{B.4.49})$$

by starting with a known solution $u(x, t)$ and perturbing its initial data by a smooth, asymptotically vanishing function, $q(x)$:

$$u^{[\epsilon]}(x, 0) = u(x, 0) + \epsilon q(x). \quad (\text{B.4.50})$$

The solution of KdV associated with these initial conditions will in general be a power series in ϵ of the form:

$$u^{[\epsilon]}(x, t) = u(x, t) + \epsilon q(x, t) + O(\epsilon^2), \quad (\text{B.4.51})$$

for some function:

$$q(x, t) = \left. \frac{du^{[\epsilon]}}{d\epsilon} \right|_{\epsilon=0}. \quad (\text{B.4.52})$$

We can write the associated solution, $\psi^{[\epsilon]}$, and eigenvalue $\tilde{\lambda}^{[\epsilon]}$, of the Schrödinger equation (2.33) as:

$$\begin{aligned}\psi^{[\epsilon]} &= \psi + \epsilon\phi + O(\epsilon^2), \\ \tilde{\lambda}^{[\epsilon]} &= \tilde{\lambda} + \epsilon \left. \frac{d\tilde{\lambda}}{d\epsilon} \right|_{\epsilon=0} + O(\epsilon^2).\end{aligned}\tag{B.4.53}$$

Substituting $\psi^{[\epsilon]}$ into (2.33) and differentiating with respect to ϵ we obtain, at $\epsilon = 0$:

$$\phi_{xx} = \left(q + \left. \frac{d\tilde{\lambda}}{d\epsilon} \right|_{\epsilon=0} \right) \psi + (u + \tilde{\lambda})\phi.\tag{B.4.54}$$

We now define the inner product of two functions in the usual way:

$$(f, g) \equiv \int_{-\infty}^{\infty} dx f(x)g(x).\tag{B.4.55}$$

Taking the inner product of (B.4.54) with ψ and assuming total derivatives vanish we obtain:

$$\begin{aligned}(\psi, \phi_{xx} - [u + \tilde{\lambda}]\phi) &= \left(\psi, q\psi + \left. \frac{d\tilde{\lambda}}{d\epsilon} \right|_{\epsilon=0} \psi \right), \\ \Rightarrow (\psi_{xx} - [u + \tilde{\lambda}]\psi, \phi) &= \left. \frac{d\tilde{\lambda}}{d\epsilon} \right|_{\epsilon=0} (\psi, \psi) + (\psi, q\psi), \\ \Rightarrow \left. \frac{d\tilde{\lambda}}{d\epsilon} \right|_{\epsilon=0} &= -(\psi^2, q),\end{aligned}\tag{B.4.56}$$

where we have assumed that the inner product of ψ with itself is unity. We have also used the fact that $Q \equiv \partial^2 - u$ is self-adjoint with respect to the inner product. Thinking of $\tilde{\lambda}$ as a functional, $\tilde{\lambda}[u]$, of $u(x, t)$, we can write:

$$\left. \frac{d\tilde{\lambda}[u^{[\epsilon]}]}{d\epsilon} \right|_{\epsilon=0} = \left. \frac{d\tilde{\lambda}[u + \epsilon q]}{d\epsilon} \right|_{\epsilon=0} = \int_{-\infty}^{\infty} dx \frac{\delta \tilde{\lambda}[u]}{\delta u(x, t)} q(x, t) = \left(\frac{\delta \tilde{\lambda}[u]}{\delta u(x, t)}, q \right),\tag{B.4.57}$$

using the usual chain rule for functional derivatives. Comparing (B.4.56) and (B.4.57) we identify:

$$\frac{\delta \tilde{\lambda}[u]}{\delta u(x, t)} = -\psi^2.\tag{B.4.58}$$

We now use the fact stated at the start of this section, that for a solitary wave, $f(x + ct)$, all of the conserved quantities are proportional to f itself. This implies that $\psi \propto f^{1/2}$, since $\tilde{\lambda}[u]$ is itself a conserved quantity. That this works can be verified by

solving the Schrödinger equation with the potential given in (2.20).

Substituting $\psi \propto f^{1/2}$ into the Schrödinger equation, multiplying through by $f^{1/2}$ and differentiating we find:

$$(\partial^2 - f - \tilde{\lambda})f^{\frac{1}{2}} = 0 \quad \Rightarrow \quad -4\tilde{\lambda}f_x = 6ff_x - f_{xxx}. \quad (\text{B.4.59})$$

Substituting $u = f(x + ct)$ into (2.11) we find:

$$-4cf_x = 6ff_x - f_{xxx}, \quad (\text{B.4.60})$$

and comparison of the proceeding two results allows us to conclude that $c = \tilde{\lambda}$; that is, the speed of the solitary wave is proportional to the eigenvalue of the Schrödinger equation. What is more, because solitary waves only travel in one direction, that is $c > 0$, we see that $\tilde{\lambda}$ is always positive. Since positive $\tilde{\lambda}$ corresponds to bound states of Schrödinger's equation, we can therefore associate solitary waves to the discrete bound state solutions of the Baker function, ψ . That is, *for each bound state eigenvalue of Schrödinger's equation there exists a soliton solution of the KdV equation.*

B.5 Inverse Scattering Theory

In Section B.4 we saw how bound states of the Schrödinger equation (2.33) correspond to the soliton solutions of the KdV equation. This forms the basis for the powerful method of solving non-linear integrable equations via the techniques of inverse scattering theory. To go through this in great detail here would be straying away from the point of this thesis somewhat, but by the same token it would be remiss not to mention it in passing. So below are a few brief paragraphs explaining the technique in cursory detail.

We begin with the Schrödinger equation³ (2.33), and consider the standard case of a plane wave, e^{ikx} , incident from the direction $x \rightarrow -\infty$, where $k = (-\tilde{\lambda})^{1/2}$. After scattering from the potential, $u(x, t)$, we expect a transmitted portion $T(k, t)e^{ikx}$ to continue towards $x \rightarrow \infty$, and a reflected portion $R(k, t)e^{-ikx}$ to return to $x \rightarrow -\infty$.

³In this section we assume that $u(x, t)$ is asymptotically vanishing.

Unitarity requires that $|R(k, t)|^2 + |T(k, t)|^2 = 1$. Mathematically we have:

$$\begin{aligned}\psi(x \rightarrow -\infty, t; k) &\sim e^{ikx} + R(k, t)e^{-ikx}, \\ \psi(x \rightarrow +\infty, t; k) &\sim T(k, t)e^{ikx}.\end{aligned}\tag{B.5.61}$$

The basic idea is to start with some initial scattering data, $R(k, 0)$ and $T(k, 0)$, corresponding to an initial potential $u(x, 0)$; then evolve the system using the Schrödinger equation to evaluate $R(k, t)$ and $T(k, t)$ at later times. We then use this evolved data to reconstruct $u(x, t)$. The key point is that the time-evolution of the scattering data is relatively simple to compute because of the linearity of the Schrödinger problem. We start by defining two *Jost* functions⁴, $f(x, k)$ and $g(x, k)$, that are solutions of Schrödinger's equation satisfying:

$$\begin{aligned}f(x, k) &\sim e^{ikx} \quad \text{as } x \rightarrow \infty, \\ g(x, k) &\sim e^{-ikx} \quad \text{as } x \rightarrow -\infty.\end{aligned}\tag{B.5.62}$$

From the Schrödinger equation we see that if $f(x, k)$ is a solution then so is its complex conjugate, which can only be $f(x, -k)$. The differing boundary conditions as $x \rightarrow \infty$ means that $f(x, k)$ and $f(x, -k)$ are linearly independent; as is true for $g(x, k)$ and $g(x, -k)$. Since the Schrödinger equation is second order we know that it has two linearly independent solutions. Therefore we can write:

$$\begin{aligned}f(x, k) &= a(k)g(x, -k) + b(k)g(x, k), \\ g(x, k) &= \tilde{a}(k)f(x, -k) + \tilde{b}(k)f(x, k),\end{aligned}\tag{B.5.63}$$

where it can be shown [12] that $a(k) = \tilde{a}(k)$ and $b(k) = -\tilde{b}(-k)$. Using the form of (B.5.61) we see that we must have:

$$\begin{aligned}\psi(x, k) &= g(x, -k) + R(k)g(x, k) = T(k)f(x, k), \\ \Rightarrow T(k) &= \frac{1}{a(k)}, \quad R(k) = \frac{b(k)}{a(k)},\end{aligned}\tag{B.5.64}$$

The crux of the inverse scattering method is to extend the argument, k , of the Jost functions to complex values of k . The Jost functions turn out to be analytic in the upper half k -plane and hence so are $a(k)$ and $b(k)$. This implies that the functions $R(k)$ and $T(k)$ can only be singular if $a(k)$ vanishes. This can be shown to only

⁴For clarity we will often suppress the t dependence of functions.

be possible if k is imaginary. These singularities are simple zeroes of $a(k)$; if we let $k = i\kappa_n$ for the n th pole, they turn out to be bound states of Schrödinger's equation with eigenvalue κ_n .

To proceed, we convert to a new set of coordinates, $(P(k), p_n, Q(k), q_n)$, given by:

$$P(k) \propto k \ln |a(k)|, \quad Q(k) \propto \arg b(k), \quad p_n \propto \kappa_n^2, \quad q_n \propto \ln |b_n|, \quad (\text{B.5.65})$$

with $b_n \equiv b(\kappa_n)$. The key result is that these variables *are just the action-angle variables of the KdV equation*. They satisfy the canonical Poisson bracket relations:

$$\{P(k), P(k')\} = 0 = \{Q(k), Q(k')\}, \quad \{Q(k), P(k')\} = \delta(k - k'). \quad (\text{B.5.66})$$

Recall that the time evolution of action angle variables is extremely simple (2.6). In the case of KdV it can be shown that the set $(P(k), p_n, Q(k), q_n)$ evolves according to:

$$\begin{aligned} \frac{dP(k, t)}{dt} &= \{P(k, t), H\} = 0, & \frac{dp(k, t)}{dt} &= \{p(k, t), H\} = 0, \\ \frac{dQ(k, t)}{dt} &= \{Q(k, t), H\} = Ak^3, & \frac{dp(k, t)}{dt} &= \{p(k, t), H\} = B\kappa_n^3, \end{aligned} \quad (\text{B.5.67})$$

where H is the Hamiltonian of the KdV hierarchy defined in (2.14). The coefficients A and B here depend on the normalisation used for the KdV and Schrödinger equations, but can be worked out. The above equations are equivalent to:

$$a(k, t) = a(k, 0), \quad b(k, t) = b(k, 0)e^{Eik^3t}, \quad b_n(t) = b_n(0)e^{F\kappa_n^3t}, \quad (\text{B.5.68})$$

for some numbers E and F , which can again be calculated. So, given the initial scattering data $a(k, 0)$, $b(k, 0)$ and $b_n(0)$ associated to an initial configuration, $u(x, 0)$, of the KdV equation, we can evolve the system in time very simply by using (B.5.68) to obtain $a(k, t)$, $b(k, t)$ and $b_n(t)$. Then we can use the Schrödinger equation to reconstruct the evolved potential $u(x, t)$. To do this we must make use of the *Marchenko* integral equation, which allows one to determine the full potential from just the asymptotic behaviour of the wavefunction [87]. So we now understand why a complicated non-linear equation can be controlled by a linear equation such as the Schrödinger equation: all along we were working in action-angle variables without realising it. All integrable systems can in principle be written in action-angle variables. Accordingly, this implies that all integrable systems should be controlled by the analogue of a Schrödinger equation.

B.6 Solitons and Supersymmetric Quantum Mechanics

In this subsection we will again study the Schrödinger problem of (2.33), but this time in the guise of supersymmetric quantum mechanics [88]. In Chapter 5, when we study the properties of FZZT brane wavefunctions in minimal superstring theory, we will make use of many of the results outlined here.

We study the following Schrödinger equation:

$$H_1\psi^{[1]}(x) = -\frac{d^2\psi^{[1]}}{dx^2} + V_1(x)\psi^{[1]}(x) = E^{[1]}\psi^{[1]}, \quad (\text{B.6.69})$$

where $V_1(x)$ is the potential, and $E^{[1]}$ the energy. Let us consider a ground state wavefunction, $\psi_0^{[1]}$, which we will assume (for now) has zero energy. Notice that once we know the ground state wavefunction (or any bound state wavefunction for that matter), we can automatically reconstruct the potential from the relationship $V_1 = \psi_{0,xx}^{[1]}/\psi_0^{[1]}$. That we can do this is the crux of the inverse scattering methods described in Section B.5. We can factorise (B.6.69) using $H_1 = a^\dagger a$. A quick calculation yields:

$$a = -\frac{d}{dx} + W(x), \quad a^\dagger = -\frac{d}{dx} - W(x), \quad (\text{B.6.70})$$

with:

$$V_1(x) = W^2(x) - W_x(x), \quad W(x) = -\frac{\psi_{0,x}^{[1]}}{\psi_0^{[1]}}. \quad (\text{B.6.71})$$

Notice that (B.6.71) is just the Miura map (2.27), and so all we have done here is to reverse the procedure that led us to the Schrödinger equation in the first instance. Notice that any wavefunction satisfying $a\psi_0^{[1]} = 0$ automatically satisfies the Schrödinger equation $H_1\psi_0^{[1]} \equiv a^\dagger a\psi_0^{[1]} = 0$. We can construct a closely related Hamiltonian, H_2 , simply by reversing the order of the operators a and a^\dagger . So we have $H_2 = aa^\dagger$. This gives another Schrödinger equation:

$$H_2\psi^{[2]}(x) = -\frac{d^2\psi^{[2]}}{dx^2} + V_2(x)\psi^{[2]}(x) = E^{[2]}\psi^{[2]}, \quad (\text{B.6.72})$$

with the potential, $V_2(x)$, given by:

$$V_2(x) = W^2(x) + W_x(x). \quad (\text{B.6.73})$$

The potentials $V_1(x)$ and $V_2(x)$ are known as *supersymmetric partner potentials*. Let us note that since the n -th bound state of H_1 , $\psi_n^{[1]}$ satisfies $H_1\psi_n^{[1]} = E_n^{[1]}\psi_n^{[1]}$ then:

$$aH_1\psi_n^{[1]} \equiv aa^\dagger a\psi_n^{[1]} = aE_n^{[1]}\psi_n^{[1]} \Rightarrow H_2(a\psi_n^{[1]}) = E_n^{[1]}(a\psi_n^{[1]}), \quad (\text{B.6.74})$$

which means that $\psi_n^{[2]} \equiv a\psi_n^{[1]}$ is a solution of (B.6.72) with the same energy. The only solutions of (B.6.69) that can not be mapped to solutions of (B.6.72) in this way are those for which $a\psi^{[1]} = 0$, and hence $H_1\psi^{[1]} = 0$. Since we are dealing with a one-dimensional problem there can only be only one such wavefunction. This is just the ground state $\psi_0^{[1]}$, which we therefore say that has no SUSY partner. The converse argument is also true, so solutions of (B.6.72) will be mapped to solutions of (B.6.69) by the a^\dagger operator, providing that they are not annihilated. So H_1 and H_2 are said to be *almost isospectral*; that is they share almost the same set of eigenvalues.

The first excited state of H_1 (with energy $E_1^{[1]}$) then maps to the ground state of H_2 ; we see that the H_2 will have one fewer bound state overall than H_1 had. We can perform the same procedure again by forming the supersymmetric partner of $\tilde{H}_2 = H_2 - E_1^{[1]}$. This will give us a new Hamiltonian, H_3 , which is almost isospectral with $\tilde{H}_1 = H_1 - E_1^{[1]}$, but having two fewer bound states. In this way we can form a whole hierarchy of almost isospectral Hamiltonians. In fact, if the Hamiltonian we start from, H_1 , has p bound states, then we can form p almost isospectral Hamiltonians from it.

We can combine the Hamiltonians H_1 and H_2 in a way that will make the supersymmetry manifest. We write a supersymmetric Hamiltonian, H , as:

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad (\text{B.6.75})$$

which acts on the column vector composed of $\psi^{[1]}$ and $\psi^{[2]}$. We now define supercharges Q and Q^\dagger as:

$$Q = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & a^\dagger \\ 0 & 0 \end{pmatrix}. \quad (\text{B.6.76})$$

The triplet H , Q and Q^\dagger then satisfies the following commutation and anticommutation relations:

$$[H, Q] = [H, Q^\dagger] = 0, \quad \{Q, Q^\dagger\} = H, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \quad (\text{B.6.77})$$

The operators Q and Q^\dagger can be thought of as operators that change bosonic degrees of freedom into fermionic ones and vice versa. The degeneracy in the spectra of H_1 and H_2 is a consequence of the fact that H commutes with the two supercharges. The

ground state of the supersymmetric theory is therefore:

$$|0\rangle = \begin{bmatrix} \psi_0^{[1]}(x) \\ 0 \end{bmatrix}, \quad (\text{B.6.78})$$

which is annihilated by both Q and Q^\dagger . If $\psi_0^{[1]}$ has non-zero energy, or is non-normalisable, then H_2 will in general have a non-zero eigenfunction, $\psi_0^{[2]}$, having the same energy as $\psi_0^{[1]}$. This will mean that the ground state of the supersymmetric theory will be:

$$|0\rangle = \begin{bmatrix} \psi_0^{[1]}(x) \\ \psi_0^{[2]}(x) \end{bmatrix}, \quad (\text{B.6.79})$$

which is not annihilated by Q and Q^\dagger . This is the origin of the above statement that these Hamiltonians have broken supersymmetry. In Chapter 5 we will see an explicit example of a hierarchy of Hamiltonians that are all interrelated via supersymmetry in the above way. These will all have non-normalisable zero energy states and hence will break supersymmetry.

The arguments above can be naturally extended to the case of Schrödinger problems with continuous parts to their spectra. In this case we have that the potentials, $V_{1,2}$, are finite as $x \rightarrow -\infty$ and/or $x \rightarrow \infty$. In the below we will assume that the potentials are finite in *both* directions, though the extension to the other cases follows trivially. We have:

$$V \rightarrow W_\pm^2 \text{ as } x \rightarrow \pm\infty, \quad \text{where } W_\pm \equiv W(x \rightarrow \pm\infty), \quad (\text{B.6.80})$$

since derivatives of $W(x)$ will be subleading. As in Section B.5, we consider the standard case of a plane wave, e^{ikx} , incident from the direction $x \rightarrow -\infty$, where $k = \sqrt{E - W_-^2}$. After scattering from $V_{1,2}$ we expect a transmitted portion $T_{1,2}(k)e^{ik'x}$ to continue towards $x \rightarrow \infty$, where $k' = \sqrt{E - W_+^2}$, and a reflected portion $R_{1,2}(k)e^{-ikx}$ to return to $x \rightarrow -\infty$.

By analogy with (B.6.70) and the discrete case, it is easy to show that:

$$\begin{aligned} e^{ikx} + R_1(k)e^{-ikx} &= N[(-ik + W_-)e^{ikx} + (ik + W_-)R_2(k)e^{-ikx}], \\ T_1(k)e^{ik'x} &= N[T_2(k)(-ik' + W_+)e^{ik'x}], \end{aligned} \quad (\text{B.6.81})$$

where N is some normalisation constant that can be eliminated to give:

$$R_1(k) = \left(\frac{W_- + ik}{W_- - ik} \right) R_2(k), \quad T_1(k) = \left(\frac{W_+ - ik'}{W_- - ik} \right) T_2(k). \quad (\text{B.6.82})$$

Using these relations we see that $|R_1|^2 = |R_2|^2$ and $|T_1|^2 = |T_2|^2$, so the SUSY partner potentials have identical reflection and transmission probabilities. We also observe that $R_1(k)$ has one extra pole in the complex k -plane in comparison to $R_2(k)$: at $k = -iW_-$. Since $k = \sqrt{E - W_-^2}$ we see that this pole corresponds to a zero energy state. The importance of this observation, and indeed the whole formalism of this subsection, will be seen again in the study of minimal string theory. Two other observations are that if $W_+ = W_-$ then $T_1(k) = T_2(k)$, and that if $W_- = 0$ then $R_1(k) = -R_2(k)$.

Appendix C

Additional Information for Chapter 3

C.1 Solving The Hermitian Matrix Model On The Sphere

Here we will calculate explicitly the free energy of the Hermitian matrix model on the sphere. We start by taking the large N limit of (3.30), whereby the position coordinate λ_i is replaced by a continuous field $\lambda(x) \equiv \lambda(i/N)$. One finds that the free energy of the planar diagrams (up to g independent constants) is given by:

$$F_M^{(0)}(g) = \int_0^1 dx \left[\frac{1}{2} \lambda^2(x) + g \lambda^4(x) \right] - \int_0^1 \int_0^1 dx dy \ln |\lambda(x) - \lambda(y)|, \quad (\text{C.1.1})$$

supplemented by the stationarity condition:

$$\frac{\delta F_M^{(0)}}{\delta \lambda} = \lambda(x) + 4g \lambda^3(x) - 2 \oint_0^1 \frac{dy}{\lambda(x) - \lambda(y)} = 0, \quad (\text{C.1.2})$$

where \oint refers to the Cauchy principal value of the integral. We proceed by introducing a density of eigenvalues $\rho(\lambda) = dx/d\lambda$, normalised according to $\int_{-2a}^{2a} d\lambda \rho(\lambda) = 1$. Here we have assumed that $\rho(\lambda)$ vanishes outside some support $(-2a, 2a)$, which must be true or else (C.1.2) will be inconsistent for large λ . The planar free energy (C.1.1)

and stationarity condition (C.1.2) become:

$$F_M^{(0)}(g) = \int_{-2a}^{2a} d\lambda \rho(\lambda) \left[\frac{1}{2} \lambda^2(x) + g \lambda^4(x) \right] - \int_{-2a}^{2a} \int_{-2a}^{2a} d\lambda d\mu \rho(\lambda) \rho(\mu) \ln |\lambda(x) - \lambda(y)|, \quad (\text{C.1.3})$$

$$\lambda(x) + 4g\lambda^3(x) = 2 \oint_{-2a}^{2a} d\mu \frac{\rho(\mu)}{\lambda - \mu}, \quad |\lambda| \leq 2a, \quad (\text{C.1.4})$$

We can encode the stationarity condition by introducing the following analytic function:

$$W(\lambda) = \int_{-2a}^{2a} d\mu \frac{\rho(\mu)}{\lambda - \mu}, \quad (\text{C.1.5})$$

defined for complex values of λ *outside* the real interval $(-2a, 2a)$. It is clear that $W(\lambda)$ has the following key properties:

- It is analytic in the complex λ plane cut along $(-2a, 2a)$.
- Since $\rho(\lambda)$ is normalised, $W(\lambda) \sim \frac{1}{\lambda}$ as $|\lambda| \rightarrow \infty$.
- It is real for real λ outside the support.
- If $\lambda_0 \in (-2a, 2a)$, $W(\lambda_0 \pm i\varepsilon) \sim \frac{1}{2}\lambda_0 + 2g\lambda_0^3 \mp i\pi\rho(\lambda_0)$ for $\varepsilon \ll 1$.

These conditions uniquely fix $W(\lambda)$ in the form:

$$W(\lambda) = \frac{1}{2}\lambda + 2g\lambda^3 - \left(\frac{1}{2} + 4ga^2 + 2g\lambda^2 \right) \sqrt{\lambda^2 - 4a^2}, \quad (\text{C.1.6})$$

$$\text{with} \quad 12ga^4 + a^2 - 1 = 0. \quad (\text{C.1.7})$$

The square root here takes the positive sign for real $\lambda > 2a$, the negative sign for real $\lambda < -2a$. This renders the function odd. The latter condition for a stems from the second requirement in the above list. The fourth requirement then yields:

$$\rho(\lambda) = \frac{1}{\pi} \left(\frac{1}{2} + 4ga^2 + 2g\lambda^2 \right) \sqrt{4a^2 - \lambda^2}, \quad |\lambda| \leq 2a. \quad (\text{C.1.8})$$

Substituting into (C.1.3) then gives:

$$F_M^{(0)}(g) - F_M^{(0)}(0) = \frac{1}{24}(a^2 - 1)(9 - a^2) - \frac{1}{2} \ln a^2. \quad (\text{C.1.9})$$

Expanding perturbatively for $a^2(g)$ using (C.1.7):

$$a^2(g) = \frac{1}{24g} [(1 + 48g)^{1/2} - 1] = 1 - 12g + 2(12g)^2 - 5(12g)^3 + O(g^4), \quad (\text{C.1.10})$$

and substituting into the above gives the following perturbative series:

$$F_M^{(0)}(g) - F_M^{(0)}(0) = 2g - 18g^2 + 288g^3 - 6048g^4 + O(g^5). \quad (\text{C.1.11})$$

The interpretation of this series is that each power of g present will correspond to an extra loop in the vacuum diagram, with the g^1 term being the two-loop diagram. The numerical factors then count the number of different possible diagrams including their symmetry factors at each order. The results of this matrix model calculation agree exactly with the answers generated by more primitive methods.

C.2 Orthogonal Polynomials and The Double Scaling Limit of The Hermitian Matrix Model

In this section we will introduce the elegant methods of orthogonal polynomials and use them to solve the double scaled Hermitian matrix model. We wish to compute the following partition function:

$$\begin{aligned} Z_M(N, g) = e^{-F_M} &= \int \mathcal{D}M \exp \left(-\frac{N}{\gamma} \text{Tr} [V(M)] \right) \\ &= \int \prod_i d\lambda_i \Delta^2(\lambda) dU \exp \left(-\frac{N}{\gamma} \sum_i V(\lambda_i) \right), \end{aligned} \quad (\text{C.2.12})$$

where we have introduced a parameter γ that we will use to tune the model to criticality¹. We define polynomials, $P_n(\lambda) = \lambda^n + \dots$, orthogonal with respect to the measure $d\beta(\lambda) \equiv d\lambda e^{-N/\gamma V(\lambda)}$:

$$\int d\beta(\lambda) P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}, \quad (\text{C.2.13})$$

which implies that they satisfy the following recursion relation:

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda), \quad (\text{C.2.14})$$

where R_n and S_n are some coefficients. It is easy to show that $R_n = h_n/h_{n-1}$, and that for odd potentials $S_n = 0$. Recall the fact that the Vandermonde determinant $\Delta(\lambda)$ can be written as the determinant of the matrix $T(N)$ (see the discussion below (3.31)). By using the property of determinants that adding one row (column) to

¹So that we can leave the parameters of the potential g_i fixed

another row (column) leaves the determinant invariant, we realise that we can write $\Delta(\lambda)$ as the determinant of another $N \times N$ matrix, $T_P(N)$, whose i, j entry is $P_{j-1}(\lambda_i)$. Breaking the determinant up into a sum of minors yields the result:

$$Z_M = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{i=1}^{N-1} R_i^{N-i} \quad \text{with} \quad h_0 = \int d\beta(\lambda), \quad (\text{C.2.15})$$

and so the problem of evaluating the partition function is reduced to that of calculating the R_i . The free energy, F_M , can be written as:

$$F_M = \sum_{k=1}^{N-1} (N-k) \ln R_k + \text{constants}. \quad (\text{C.2.16})$$

Using the recursion relation and the orthogonality condition we can write:

$$\int d\beta(\lambda) P_n \frac{dP_n}{d\lambda} = 0 \quad \Rightarrow \quad \int d\beta(\lambda) V'(\lambda) P_n^2 = 0, \quad (\text{C.2.17})$$

$$\int d\beta(\lambda) P_{n-1} \frac{dP_n}{d\lambda} = n h_{n-1} \quad \Rightarrow \quad \int d\beta(\lambda) V'(\lambda) P_{n-1} P_n = \frac{\gamma n h_{n-1}}{N}, \quad (\text{C.2.18})$$

where we have integrated by parts to get the equations on the right-hand side. The first equation here is trivially satisfied if the potential is odd ($S_n = 0$). Specialising to our earlier quartic potential, $V = \lambda^2 + g\lambda^4$, we find that (C.2.18) yields the recursion relation:

$$\frac{\gamma n}{N} = R_n(g) (1 + 4g [R_{n-1}(g) + R_n(g) + R_{n+1}(g)]). \quad (\text{C.2.19})$$

In the large N limit we find that R_n approaches a continuous function $R(x)$ of the variable $x = n/N$. So, ignoring $1/N$ corrections, we can write the recurrence relation as:

$$\gamma x = R(x, g) + 12g R^2(x, g), \quad (\text{C.2.20})$$

$$\begin{aligned} \Rightarrow R(x, g) &= \frac{1}{24g} [(1 + 48g\gamma x)^{1/2} - 1] \\ &= 1 - 12g\gamma x + 2(12g\gamma x)^2 - 5(12g\gamma x) + \dots \end{aligned} \quad (\text{C.2.21})$$

Substituting into the continuum version of (C.2.16) gives:

$$F_M(g) = N^2 \int_0^1 dx (1-x) \ln R(x), \quad (\text{C.2.22})$$

and setting $\gamma = 1$ we find:

$$F_M^{(0)}(g) - F_M^{(0)}(0) = 2g - 18g^2 + 288g^3 - 6048g^4 + O(g^5), \quad (\text{C.2.23})$$

which is just the same result we had before (C.1.11). Now however, it is very easy in principle to continue the above calculation to include the powers of $1/N$ corrections. This will yield free energy of higher surfaces such as the torus. However, by taking the double scaling limit (3.45) explained above, we can do far better than that. Actually we need to use a modified version of (3.45) to take into account the new parameter γ . We set $g = g_c = -1/48$, which makes the critical value of γ equal to unity, and then use the scaling ansatz:

$$\gamma = 1 - \mu\delta^4, \quad \frac{1}{N} = \nu\delta^5, \quad (\text{C.2.24})$$

We now define a new variable z via $\gamma x = 1 - z\delta^4$. At criticality we see from (C.2.20) that $R_c = 2$. We therefore set $R = 2 + u(z)\delta^k$ for some power k that we will determine. This gives:

$$\begin{aligned} F_M(\mu) &= -\frac{1}{\nu^2\delta^{10}} \int_{\mu}^{1/\delta^4} dz (\mu - z) \delta^8 \ln(2 - u(z)\delta^k) + \dots \\ \therefore F_M(\mu) - F_M(0) &= -\frac{1}{\nu^2} \int_0^{\mu} dz (\mu - z) u(z) \delta^{k-2} + \dots, \end{aligned} \quad (\text{C.2.25})$$

the only way this can be finite in the $\delta \rightarrow 0$ limit is if $k = 2$. The function $u(z)$ then satisfies the following relationship:

$$u(z) = -\nu^2 \frac{\partial^2 F_M}{\partial z^2}, \quad (\text{C.2.26})$$

as can be seen by substituting into (C.2.25) and integrating by parts. Performing the same scaling limit on (C.2.19) we obtain, at leading order in δ , the result:

$$-\frac{\nu^2}{3} \frac{\partial^2 u}{\partial z^2} + u^2 - z = 0, \quad (\text{C.2.27})$$

where we have taken the liberty of rescaling the variables. This equation is known as the *string equation*, and is also the Painlevé I equation of the mathematical literature.

C.3 The Unitary Matrix Model and Type 0B Minimal Superstring Theories

In [89–91] unitary matrix models were studied. The partition functions of these models have the form:

$$Z = \int dU \exp \left(-\frac{N}{\gamma} \text{Tr} V(U + U^\dagger) \right), \quad (\text{C.3.28})$$

where U is a unitary $N \times N$ matrix. Just as in the case of the Hermitian and complex matrix models, one can double scale this partition function to various multicritical points to yield associated string equations. The simplest relevant critical point has a quadratic potential and is labelled by $n = 2$. Similarly, we can label a cubic potential by $n = 3$, a quartic by $n = 4$, etc. The resultant string equations for the n -th model are given in terms of differential polynomials, $F_\ell[f, g]$ and $G_\ell[f, g]$, for functions $f(z)$ and $g(z)$:

$$\sum_{\ell=0}^n t_\ell F_{\ell+1} = 0 = \sum_{\ell=0}^n t_\ell G_{\ell+1}, \quad (\text{C.3.29})$$

where the \tilde{t}_n are parameters analogous to those in the Hermitian and complex cases, but with $t_0 \equiv z$. The free energy is then given by:

$$F'' = \frac{1}{4}(f^2 + g^2). \quad (\text{C.3.30})$$

It turns out that the differential polynomials F_ℓ and G_ℓ are exactly those of the Zakharov-Shabat hierarchy given in (B.3.43). Let us change variables to $f = r \cos \theta$, $g = r \sin \theta$ and take linear combinations of the two string equations in (C.3.29). Then, using H_ℓ defined in (B.3.40), we can write:

$$\sum_{\ell=0}^n t_\ell (f F_{\ell+1} + g G_{\ell+1}) \equiv \sum_{\ell=0}^n t_\ell r R_{\ell+1} = 0, \quad (\text{C.3.31})$$

$$\sum_{\ell=0}^n t_\ell (f G_{\ell+1} - g F_{\ell+1}) = \sum_{\ell=0}^n t_\ell H'_{\ell+1} = 0. \quad (\text{C.3.32})$$

Now, using the fact that $H_1 = 0$, we arrive at our final string equations and free energy relation:

$$\sum_{\ell=0}^n t_\ell R_{\ell+1} = 0 = \sum_{\ell=0}^n t_\ell H_{\ell+1} + i\Gamma, \quad F'' = \frac{r^2 - z}{4}, \quad (\text{C.3.33})$$

where Γ is an integration constant. The first few differential polynomials are given by:

$$\begin{aligned} R_0 &= 0, & H_0 &= 1, \\ R_1 &= r, & H_1 &= 0, \\ R_2 &= \omega r, & H_2 &= -\frac{r^2}{2}, \\ R_3 &= r'' - r\omega^2 - \frac{1}{2}r^3, & H_3 &= r^2\omega, \end{aligned} \quad (\text{C.3.34})$$

where $\omega \equiv \theta'$. As in the Hermitian and complex cases, we find that we can absorb \tilde{t}_{m-1} into a boundary operator by adding a constant to θ and redefining the lower \tilde{t}_k . In [31] convincing evidence was presented that these string equations describe the Type 0B $(2, 2n)$ string theories. Recall from Section 3.3 that for n odd the theory is not modular invariant. However, despite this problem we see that these odd- n models are still defined by the Zakharov-Shabat hierarchy. The simplest model is again $(2, 4)$ pure supergravity, with the following string equation obtained by eliminating ω from (C.3.33):

$$r'' - \frac{1}{2}r^3 + \frac{1}{2}zr + \frac{\Gamma^2}{r^3} = 0, \quad (\text{C.3.35})$$

which for $\Gamma = 0$ is known as the Painlevé II equation. Solving this equation we find the following large z asymptotics:

$$\begin{aligned} F'' &= \frac{z}{4} + \frac{\nu^2(4\Gamma^2 - 1)}{4z^2} - \frac{\nu^4(4\Gamma^2 - 1)(4\Gamma^2 - 9)}{8z^5} + \dots \quad \text{as } z \rightarrow +\infty, \\ F'' &= \frac{i\nu\Gamma\sqrt{2}}{4z^{1/2}} - \frac{\nu^2\Gamma^2}{4z^2} + \frac{5i\nu^3\Gamma(4\Gamma^2 + 1)\sqrt{2}}{64z^{7/2}} + \dots \quad \text{as } z \rightarrow -\infty, \end{aligned} \quad (\text{C.3.36})$$

which, after a rescaling and sign change of z , is identical to the Type 0A expansion given by (3.68) up to analytic (non-universal) terms. This is a peculiarity of the $\hat{c} = 0$ theory and is due to the fact that the operator content of Type 0A theory for one sign of μ ($= z$) is the same as that in the Type 0B theory with the opposite sign of μ , because the only Ramond operator is the ground state². Specifically, we can get from (3.67) to (C.3.35) with $\mathcal{R} = u - z$ by applying the transformation $u(z) = h(z)^2 + z$ to the latter equation, and then rescaling $h(z) \mapsto 2^{-1/6}r(z)$, $z \mapsto -2^{-1/3}z$. For other minimal models the Type 0A and Type 0B partition functions will be distinct.

²The only worldsheet fermions in the problem are the Liouville fermions, ψ and $\bar{\psi}$, so going from 0A to 0B is equivalent to $\psi \mapsto -\psi$ in this model. This can be achieved by simply sending $\mu \mapsto -\mu$.

Appendix D

Additional Information for Later Chapters

D.1 Derivation of The Explicit Type 0A Bäcklund Transformation

Here we will review and extend the calculation of [56] to achieve an explicit relation that changes Γ by an integer. To start we assume $u(z)$ satisfies the usual string equation (3.67). Then, we *define* $v(z)$ via:

$$X_{\pm}[u, v] \equiv \frac{1}{2}\mathcal{R}'[u] \pm \nu\Gamma + v(z)\mathcal{R}[u] = 0, \quad (\text{D.1.1})$$

which implies:

$$v = -\frac{\frac{1}{2}\mathcal{R}'[u] \pm \nu\Gamma}{\mathcal{R}[u]}. \quad (\text{D.1.2})$$

A short calculation shows that the following equation holds:

$$\begin{aligned} 0 &= X_{\pm}^2 \mp \nu\Gamma X_{\pm} - \mathcal{R}[u]X'_{\pm} \\ &\equiv (v^2 - v')\mathcal{R}^2[u] - \frac{1}{2}\mathcal{R}[u]\mathcal{R}''[u] + \frac{1}{4}(\mathcal{R}'[u])^2 - \nu^2\Gamma^2, \end{aligned} \quad (\text{D.1.3})$$

Comparison with (3.67) allows us to deduce that the inverse transformation between $u(z)$ and $v(z)$ is just the Miura map, $u = v^2 - v'$, as we would anticipate. Furthermore, we can substitute into (D.1.2) into (D.1.3) to obtain the following equation satisfied by $v(z)$:

$$\sum_{m=1}^{\infty} \left(m + \frac{1}{2}\right) \tilde{t}_m S_m[v(z)] - zv(z) = \nu C, \quad (\text{D.1.4})$$

where the differential polynomials $S_m[v]$ are given by:

$$S_m \equiv \frac{1}{2} R'_m[v^2 - v'] + v R_m[v^2 - v'], \quad (\text{D.1.5})$$

and the constant C is equal to $\frac{1}{2} \pm \Gamma$. Equation (D.1.5) provides higher order generalisations of the Painlevé II equation, which is the $m = 1$ case.

Since the transformation between $u(z)$ and $v(z)$ is true for both values of C , we find that a given $u_\Gamma(z)$ yields two functions $v_C(z)$ and $v_{1-C}(z)$, which must be related by:

$$v_C^2 - v'_C = v_{1-C}^2 - v'_{1-C}, \quad (\text{D.1.6})$$

and hence:

$$v_{1-C} = v_C - \frac{\nu(2C - 1)}{\mathcal{R}[v_C^2 - v'_C]}. \quad (\text{D.1.7})$$

On the other hand, the string equation (D.1.4) has the symmetry $v_{-C} = -v_C$, and so we obtain a result which relates $v(z)$ at one value of C to another for $C \pm 1$:

$$v_{C \pm 1} = -v_C + \frac{\nu(2C \pm 1)}{\mathcal{R}[v_C^2 \pm v'_C]}. \quad (\text{D.1.8})$$

This concludes the summary of the results derived in [56].

However, (D.1.8) implies a similar transformation for $u(z)$, which relates Γ and $\Gamma \pm 1$. Let us start with (D.1.6), which changes v_C to v_{1-C} , which we have seen is $-v_{C-1}$. Substituting in $C = \frac{1}{2} \pm \Gamma$, we see that v_{C-1} corresponds to $v_{-1/2 \pm \Gamma}$, which can be written as $v_{1/2 \pm (\Gamma \mp 1)}$. So we see that starting from a solution with $C = \frac{1}{2} + \Gamma$ will allow us to construct $u_{\Gamma-1}$; whereas starting from a solution with $C = \frac{1}{2} - \Gamma$ will allow us to construct $u_{\Gamma+1}$. Accordingly we rewrite (D.1.6) as:

$$v_{\Gamma \pm 1} = -v_\Gamma + \nu \frac{(1 \mp 2\Gamma) - 1}{\mathcal{R}[v_\Gamma^2 - v'_\Gamma]} = -v_\Gamma \mp \frac{2\nu\Gamma}{\mathcal{R}[u_\Gamma]}, \quad (\text{D.1.9})$$

where we have used the Miura map $u_\Gamma = v_\Gamma^2 - v'_\Gamma$. Writing $\mathcal{R}[u_\Gamma] = \mathcal{R}$, we now use (D.1.2) to give:

$$v_{\Gamma \pm 1} = \frac{\frac{1}{2}\mathcal{R}' \mp \nu\Gamma}{\mathcal{R}}. \quad (\text{D.1.10})$$

Differentiating this result gives:

$$v_{\Gamma \pm 1} = \frac{\frac{1}{2}\mathcal{R}''}{\mathcal{R}} - \frac{(\frac{1}{2}\mathcal{R}' \mp \nu\Gamma)\mathcal{R}'}{\mathcal{R}^2}, \quad (\text{D.1.11})$$

whereas squaring it gives:

$$v_{\Gamma\pm 1}^2 = \frac{\frac{1}{4}(\mathcal{R}')^2 \mp \nu\Gamma\mathcal{R}' + \nu^2\Gamma^2}{\mathcal{R}^2}. \quad (\text{D.1.12})$$

Now we can again use the Miura map to construct $u_{\Gamma\pm 1}$:

$$\begin{aligned} \mathcal{R}^2 u_{\Gamma\pm 1} &= \mathcal{R}^2(v_{\Gamma\pm 1}^2 - v'_{\Gamma\pm 1}) \\ &= \frac{1}{4}(\mathcal{R}')^2 \mp \nu\Gamma\mathcal{R}' + \nu^2\Gamma^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{2}(\mathcal{R}')^2 \mp \nu\Gamma\mathcal{R}'. \end{aligned} \quad (\text{D.1.13})$$

Collecting terms together then gives the final result:

$$u_{\Gamma\pm 1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\Gamma\mathcal{R}' + 4\nu^2\Gamma^2}{4\mathcal{R}^2}, \quad (\text{D.1.14})$$

which is an explicit relation between u_{Γ} and $u_{\Gamma\pm 1}$ as we required.

D.2 Scale Invariance and The String Equation

Let us further elaborate on the relationship between the *flow* equations of the KdV hierarchy, (3.59), and the $(2, 4m)$ Type 0A superstring theories by studying the behaviour of solutions, $u(x, t)$, of the KdV hierarchy under a change of scale. This connection was first realised in [32]. Recall that since the equations of the KdV hierarchy are in involution, we can always think of u as being a solution of all the equations of the hierarchy at the same time: $u(x, t_0, t_1, \dots, t_{m-1})$. Study of (B.2.24) tells us that for $u(x)$ to be invariant under $x \mapsto sx$, it must have the scaling dimension $[u] = -2[x]$; similarly, $R_n[u]$ must have the scaling dimension $[R_n] = -(2n + 3)[x]$; t_k has the scaling dimension $[t_k] = (2k + 1)[x]$. Under infinitesimal scalings, $s = 1 + \epsilon$ ($\epsilon \ll 1$) we know that a quantity f scales like $df = (s^{[f]} - 1)f \approx \epsilon[f]f$. We can write:

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \sum_{k=0}^{m-1} \frac{\partial u}{\partial t_k} dt_k, \\ \Rightarrow \quad \epsilon[u]u &= \epsilon[x]xu' + \sum_{k=0}^{m-1} \epsilon[t_k] \frac{t_k}{\alpha_k} R'_{k+1}, \\ \Rightarrow \quad -2u &= xu' + \sum_{k=1}^m (2k + 1) \frac{t_k}{\alpha_k} R'_{k+1}. \end{aligned} \quad (\text{D.2.15})$$

Now, recalling the KdV recursion relation (3.58), and defining $\tilde{t}_k \equiv (4k+2)t_k/\alpha_k$, we can rewrite the above in the form:

$$\begin{aligned}
 4u + 2xu' + \sum_{k=1}^m \tilde{t}_k (\partial^3 - 4u\partial - 2u') R_k &= 0, \\
 \Rightarrow (\partial^3 - 4u\partial - 2u') \left[\sum_{k=1}^m \tilde{t}_k R_k - x \right] &= 0, \\
 \Rightarrow (\partial^3 - 4u\partial - 2u') \mathcal{R}[u] &= 0.
 \end{aligned} \tag{D.2.16}$$

Multiplying by $\mathcal{R}[u]$, integrating and making the substitution $x \mapsto z$ we again find the Type 0A string equation [32]:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2, \tag{D.2.17}$$

where Γ is an integration constant. So we see that the Type 0A string equations and the KdV flow equations are more than just superficially related. Specifically, solutions of (D.2.17) *are* solutions of the KdV hierarchy, albeit special scale invariant ones. We have seen this in a more explicit form in Chapter 5.

